APPENDIX II

LOCAL PUBLIC GOODS IN A MORE GENERAL MODEL

In this appendix the analysis in Chapter III is extended to the case of two factors of production. It is assumed that there is more than one kind of consumer goods and that land as well as labour is used in producing the consumer goods. The production function of the *i*-th consumer good is written as

$$F^i(L_i, H_i) \qquad i = 1, \dots, k \tag{1}$$

where L_i and H_i are respectively labour and land inputs. Assuming that the production function is homogeneous of degree one, we obtain the per-unit-land production function, $f^i(l_i)$:

$$F^i(L_i, H_i) = H_i F^i(\frac{L_i}{H_i}, 1) = H_i f^i(l_i) \tag{2}$$

where l_i is the labour-land ratio, L_i / H_i.

The utility function of city residents is

$$u(z(x), h(x), X), \tag{3}$$

where $z(x) = (z_1(x), \dots, z_i(x), \dots, z_k(x)),$ is the vector of consumer goods.

In contrast to our procedure in Chapter III, we assume a vector of transportation costs $t(x) = (t_1(x), \dots, t_i(x), \dots, t_k(x)),$ for consumer goods within a city. Each city now has a port, or perhaps a railroad station at the center, where goods are bought at prices $p = (p_1, \dots, p_i, \dots, p_k)$ for distribution throughout the economy. Cities are small, so that prices are effectively parametric, and producers at x face the net price vector

$$p(x) = p - t(x). \tag{4}$$

If good *i* is produced at *x*, we obtain the following equations by profit maximization:

$$P_i(x) f^{i\prime}(l_i(x)) = w(x) \tag{5}$$

$$P_i(x)[f^i - l_i(x) f^{i\prime}(l_i(x))] = R(x) \tag{6}$$

where $w(x)$ and $R(x)$ are respectively wage rate and land rent at x.

We assume that there is also a retail market at the center of the city. In buying the consumer goods, residents in the city are assumed to incur transportation costs from the market to the place of residence. Therefore, households living at x face the price vector of the consumer good:

$$q(x) = p + t(x). \tag{7}$$

In each city one developer collects land rent and pays the rural rent and the costs of the public good. The profit is distributed equally among all households in the economy. If we assume that there are many identical cities, a household receives dividends from many developers and a change in one city does not significantly affect the total dividend, s, that a household receives. A household working at x' receives the wage, $w(x')$, and the dividend. If the household lives at x, the budget constraint is

$$w(x') + s = q(x) \cdot z(x) + \left[t_h(x) - t_h(x')\right] + R(x)h(x) \tag{8}$$

where $t_h(x)$ is the commuting costs from x to *0* and hence $t_h(x) - t_h(x')$ is the commuting costs from x to x'.

We assume that all households have the same skill and the same utility function. Then all households receive the same utility level in equilibrium. This implies that all households living at the same location must receive the same net income after commuting costs wherever they work. Therefore, we obtain

$$w(x') = w - t_h(x'), \tag{9}$$

where $w \equiv w(0)$.

A household's utility maximization yields

$$\frac{u_h}{u_{z_i}} = \frac{R(x)}{q_i(x)} \qquad I = 1, \ldots, k. \tag{10}$$

Using (9), we can rewrite (5) and (8) as

$$p_i(x) f^{i\,\prime}(l_i(x)) = w - t_h(x) \tag{11}$$

$$w + s = q(x) \cdot z(x) + t_h(x) + R(x)h(x) \tag{12}$$

Totally differentiating (11) and (6), we obtain

$$dw = p_i(x) f^{i\,\prime\prime}(l_i(x)) dl_i(x) \tag{13}$$

$$- p_i(x) l_i(x) f^{i\,\prime\prime}(l_i(x)) dl_i(x) = dR(x). \tag{14}$$

Combining these two equations, the following simple relationship can be obtained:

$$dR(x) = -l_i(x)dw \tag{15}$$

Totally differentiating (3) and (12), and noting the small city assumption that the utility level is given, we obtain

$$0 = u_z dz(x) + u_h dh(x) + u_x dX$$

$$dw = q(x) \cdot dz(x) + R(x)dh(x) + h(x)dR(x).$$

From these two equations we have

$$h(x)dR(x) = \frac{u_x}{u_z}dx + dw. \tag{16}$$

From (15) and (16) , the change of the total rent in the city due to an increase of the public good is equal to the social benefit of the public good:

$$\begin{aligned} &\int_0^{\bar{x}} \frac{dR(x)}{dx} q(x)dx \\ &= \sum_{i=1}^{k} \int -\frac{dw}{dx} l_i(x)q(x)dx + \int_{\underline{x}}^{\bar{x}} \frac{dw}{dx} N(x)dx + \int_{\underline{x}}^{\bar{x}} \frac{u_x}{u_z} N(x)dx \\ &= \frac{dw}{dx}\left[\int_{\underline{x}}^{\bar{x}} N(x)dx - \sum_{i=1}^{k} \int L_i(x)dx\right] + \int_{\underline{x}}^{\bar{x}} \frac{u_x}{u_z} N(x)dx \\ &= \int_{\underline{x}}^{\bar{x}} \frac{u_x}{u_z} N(x)dx . \end{aligned} \tag{17}$$

The last equality is obtained using the fact that the total labour force must be equal to the population of the city. Thus, even if there are more than one factor of production and more than one consumer good, the benefit of the public good is reflected in the increase of land rent in a small city.

We can also see that the profit maximization of a city developer leads to an efficient supply of the public good. A city developer maximizes

$$\int_0^{\bar{x}} [R(x) - R_a] q(x)dx - C(x),$$

where $C(X)$ is the cost of producing the public good. Then

$$\begin{aligned} &\frac{d}{dx}\left\{\int_0^{\bar{x}} [R(x) - R_a] q(x)dx - C(x)\right\} \\ &= \int_0^{\bar{x}} \frac{dR(x)}{dX} q(x)dx - C'(x) \\ &= \int_{\underline{x}}^{\bar{x}} \frac{u_x}{u_z} N(x)dx - C'(x) = 0. \end{aligned}$$

As in section 3 of Chapter III it can be seen that the last equality is the condition for an efficient supply of the public good.

Notice that this result does not depend on the number of commodities produced in the city, or on whether different goods are produced in different zones. We used only the conditions for a small city: given utility level; given price vector of consumer goods; and constant returns to scale in production. Although in general the wage rate changes as the supply of the public good changes, it does not affect the conclusion, since the effects on the production side and the consumption side cancel out each other.

This result can be interpreted in the same way as in section 1 of Chapter III. The benefits of the public good must accrue to somebody or become a deadweight loss. But there is no deadweight loss if there are no distortions in the rest of the economy. Therefore, all the benefit must be received by somebody. By the assumption of a small city, the residents cannot benefit from the public good. Because of constant returns to scale there is no profit in equilibrium. Thus the land rent is the only place the benefit appears.

This argument suggests that if returns to scale are constant, the sum of land rent and the profits (or losses) of producers reflects the benefit of the public good. It is not difficult to show that this is indeed true.