APPENDIX III

THE ENVELOPE PROPERTY

Optimization imposes a very strong structure on the problem considered. This is the reason why neoclassical economics, which assumes optimizing behaviour, has been the most successful of social sciences. One of its important aspects is the envelope property discussed in this Appendix.

The envelope property is concerned with the rate of change of the maximum (or minimum) value of a criterion function caused by a change in some parameter; for example, a change in the maximum utility level of a household caused by a change in income, a change in the minimum cost of production caused by a change in the output level, and so on. A change in a parameter in general induces a change in the optimum levels of choice variables. According to the envelope property, however, the induced change in the choice variables may be ignored in calculating the effect of a change in a parameter on the maximum value if the change is very small. In other words, a change in the maximum value caused by a marginal change in a parameter, which also induces a change in the choice variables, is equal to a change in criterion function with choice variables fixed.

In section 1, the envelope property is explained in the simplest possible case. The Envelope Theorem is stated and proved in section 2. In section 3 properties of the indirect utility function and the expenditure function are derived as applications of the Envelope Theorem.

1. The Simplest Case

The essence of the envelope property may be explained using the following simple maximization problem. Consider the problem of maximizing the criterion function, $V(x,b)$, with respect to x for a given parameter b. An interior maximum is obtained at the point where the derivative of the criterion function with respect to x is zero,

$$\frac{\partial V(x,b)}{\partial x} = 0.$$

The maximizing value of x changes as the parameter b changes: from x^* to $x^{*\prime}$ as b moves to b'. *The envelope property states that the total effect of an infinitesimal change in the parameter on the maximized value of the criterion function (including the effect of an induced change in the optimum value of x) equals the partial effect on the criterion function with the level of x fixed.* In Figure 1 the former is the movement from V to V'; and the latter from V to $\tilde{V}$. Since the criterion function is

approximately flat near the optimum point, the difference between the two, $\tilde{V}V'$, is very small compared with $V\tilde{V}$. As the change in the parameter approaches zero, the difference becomes negligible and the envelope property can be invoked.
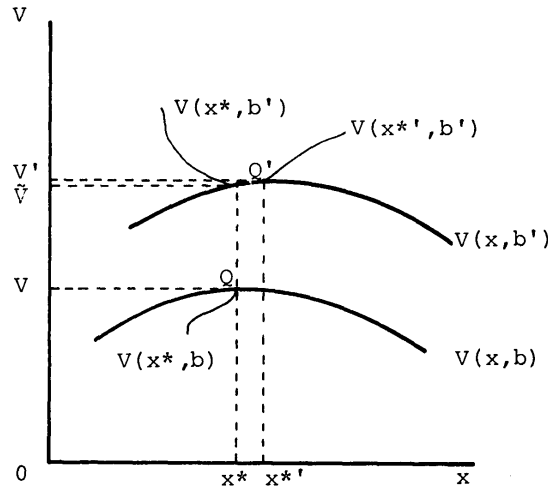

Figure 1.

The envelope property can be derived by mechanically differentiating the criterion function at the maximum. Since the optimum value of x depends on b, it can be described as a function, $x^*(b)$, of b. Then the total effect including a change in x^* is

$$\frac{dV(x^*(b),b)}{db} = \frac{\partial V}{\partial x}\frac{dx^*}{db} + \frac{\partial V}{\partial b}$$

and the partial effect is

$$\frac{\partial V(x^*,b)}{\partial b}.$$

The two are equal since $\partial V/\partial x = 0$ at the optimum.

Figure 2 illustrates why this property is called the envelope property. The heavy curve represents the maximum value, $V^*(b) = V(x^*(b),b)$, of the criterion function corresponding to different values of the parameter. The lighter curves describe the value of the criterion achieved with fixed values, $\bar{x}$ (and $\bar{x}'$) of x, as b is varied. The values and the slopes of the two types of curves, $V(x^*(b),b)$ and $V(\bar{x},b)$, are equal at the value of b for which x is optimal, that is, where $\bar{x} = x^*(b)$. The two curves are tangent at that point, and $V(\bar{x},b)$ is below $V^*(b)$ everywhere else, since

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$V*(b)$ is the maximum: $V*(b) > V(\bar{x},b)$ if $\bar{x} = x*(b)$. This holds for any $\bar{x}$ and the curve $V*(b)$ is the envelope of the curves $V(\bar{x},b)$.
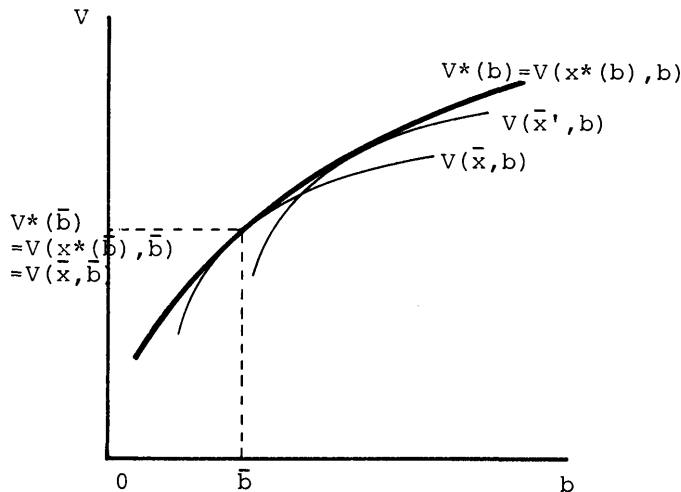

Figure 2.

Figure 2 suggests another way of proving the envelope property. Since $V*(b)$ is maximum, $V*(b) \geq V(\bar{x},b)$ for any b and $V*(b) = V(\bar{x},b)$ if $\bar{x} = x*(b)$. This implies that $V(\bar{x},b)$ lies below $V*(b)$ everywhere and the two coincide at the value of b for which $\bar{x}$ is optimal. If the two curves are smooth, this is possible only when the two curves are tangent at this point, which proves the envelope property: $dv*(b)/db = \partial V(\bar{x},b)/\partial b$ if $\bar{x} = x*(b)$.

The envelope property appears in many areas of economics. Probably the most famous application is the relationship between the long-run cost curve and the short-run cost curve. The short-run cost curve is obtained when only a subset of factors are optimally chosen, and the long-run cost curve when all factors are chosen optimally. In the short run some factor inputs are fixed whereas in the long run they become variable and can be chosen optimally. Cost curves describe the minimized cost as functions of the output. The argument in the last proof of the envelope property can be applied to show that the long-run cost curve is an envelope of short-run cost curves. [1]

Another important example is concerned with benefits of a public good. Consider a household with the utility function, $u(z,h,X)$, where z is the composite consumer good and the numeraire, h is the lot size, and X is the supply of a public good. For a given consumption bundle the marginal benefit of the public good is $\partial u(z,h,X)/\partial X$. When the consumption bundle is optimally chosen, the maximum utility level depends on the income, I, the land rent, R, and the level of the public good,

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[1] See Dixit (1976).

X, and it can be described by the indirect utility function, $v(R,I,X)$. For the optimum consumption bundle the marginal benefit of the public good is $\partial v(R,I,X)/\partial X$. The envelope property implies that $\partial u(z,h,X)/\partial X = \partial v(R,I,X)/\partial X$ if z and h are optimal given R, I, and X. This result is used in Chapter III.

2. The Envelope Theorem

Consider the problem of maximizing the criterion function $f(x,b)$ subject to the constraints $g_j(x,b) = 0$, $j = 1,2,\ldots,m$, with respect to the vector $x = (x_1,x_2,\ldots,x_n)$ for a fixed vector of parameters $b = (b_1,b_2,\ldots,b_q)$. Let $x^*(b)$ be the optimal choice for this problem. Then granted a certain regularity condition[2] there exists the vector of Lagrange multipliers $\boldsymbol{l} = (\boldsymbol{l}_1,\boldsymbol{l}_2,\ldots,\boldsymbol{l}_m)$ such that $x^*(b)$ maximizes the Lagrangian

$$\Phi(x,\boldsymbol{l},b) = f(x,b) + \boldsymbol{l} \cdot g(x,b) \tag{2.1}$$

without any constraint, where $g(x,b) = (g_1(x,b), g_2(x,b),\ldots,g_m(x,b))$, and the dot between   and $g(x,b)$ denote the inner product so that

$$\boldsymbol{l} \cdot g(x,b) = \sum_{j=1}^{h} \boldsymbol{l}_j g_j(x,b).$$

If $f(x,b)$ and $g(x,b)$ are differentiable with respect to x, the optimal choice, $x = x^*(b)$, satisfies the first order necessary conditions,

$$\partial f(x,b)/\partial x_i + \boldsymbol{l} \cdot \partial_g(x,b)/\partial x_i = 0, \qquad i = 1,2,\ldots,n. \tag{2.2}$$

The *Envelope Theorem* describes a relationship between the *maximum value* function $f^*(b) = f(x^*(b),b)$ and the Lagrangian $\Phi(x,\boldsymbol{l},b)$.

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[2] The condition is called the *Jacobian condition*, and requires that the Jacobian matrix of first order partial derivatives of constraint functions,

$$\begin{bmatrix} \dfrac{\partial g_1}{\partial x_1}, & \dfrac{\partial g_1}{\partial x_2}, & \ldots, & \dfrac{\partial g_1}{\partial x_n} \\ \dfrac{\partial g_2}{\partial x_1}, & \dfrac{\partial g_2}{\partial x_2}, & \ldots, & \dfrac{\partial g_2}{\partial x_n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \dfrac{\partial g_m}{\partial x_1}, & \dfrac{\partial g_m}{\partial x_2}, & \ldots, & \dfrac{\partial g_m}{\partial x_n} \end{bmatrix}$$

be of full row rank m at the optimum. In nonlinear programming which deals with the more general case which includes inequality constraints, a similar condition, called the *constraint qualification*, must be satisfied.

The Envelope Theorem: Assume that $f*(b)$ and $\Phi(x, \boldsymbol{l}, b)$ are continuously differentiable in b. Then at $x = x*(b)$,

$$\partial f*(b)/\partial b_k = \partial \boldsymbol{f}(x, \boldsymbol{l}, b)/\partial b_k, \qquad k = 1, 2, \ldots, q. \tag{2.3}$$

Proof

Since $x*(b)$ satisfies the constraint $g(x*(b), b) = 0$ for any b, we have

$$\sum_{i=1}^{n} (\partial g/\partial x_i)(\partial x_1^*(b)/\partial b_k) + \partial g/\partial b_k = 0, \qquad k = 1, 2, \ldots, q. \tag{2.4}$$

By the definition of the maximum value function and the first order condition (2.2), we obtain

$$\partial f*(b)/\partial b_k = \sum_{i=1}^{n} (\partial f/\partial x_i)(\partial x_1^*/\partial b_k) + \partial f/\partial b_k$$

$$= -\boldsymbol{l} \cdot \sum_{i=1}^{n} (\partial g/\partial x_i)(\partial x_1^*/\partial b_k) + \partial f/\partial b_k \tag{2.5}$$

(2.4) now yields the desired result:

$$\partial f*(b)/\partial b_k = \boldsymbol{l} \cdot \partial g/\partial b_k + \partial f/\partial b_k$$
$$= \partial \boldsymbol{f}(x, \boldsymbol{l}, b)/\partial b_k.$$

Q.E.D.

3. Applications: Properties of Indirect Utility Function and the Expenditure Function

Consider a consumer with a utility function, $u(x)$, where x is the consumption vector, $x \equiv (x_1, x_2, \ldots, x_n)$. The consumer maximizes the utility function subject to the budget constraint,

$$p \cdot x = I, \tag{3.1}$$

where p is the price vector, $p \equiv (p_1, p_2, \ldots, p_n)$, I the money income, and

$$p \cdot x = \sum_{i=1}^{n} p_i x_i.$$

The Lagrangian for this maximization problem is

$$\Psi = u(x) + \boldsymbol{d}[I - p \cdot x], \qquad\qquad (3.2)$$

where $\qquad$ is the Lagrange multiplier associated with the budget constraint (3.1). The first order conditions are

$$\partial u / \partial x_i = \boldsymbol{d} p_i, \qquad\qquad i = 1, 2, \ldots, n \qquad\qquad (3.3)$$

The optimal consumption depends on income and prices, and can be written as $x(p, I)$. Substituting $x = x(p, I)$ into the utility function $u(x)$ yields the maximum utility level, $v(p, I) \equiv u(x(p, I))$, which can be achieved at the given values of income and prices. $v(p, I)$ is called the *indirect utility function*. The Envelope Theorem, (2.3), may then be applied to examine the effect of a change in prices and the income on the maximized utility level:

$$\partial v / \partial p_i = -\boldsymbol{d} x_i, \ i = 1, 2, \ldots, n, \qquad\qquad (3.4)$$

$$\partial v / \partial I = \boldsymbol{d} \ . \qquad\qquad (3.5)$$

The latter equality shows that the Lagrange multiplier equals the marginal contribution to the maximum utility level made by an increase in income, or the marginal utility of income. The multiplier is, therefore, interpreted as the *shadow value* of the monetary income in utility terms.

If a dollar increase in income is all spent on good *i*, the increase in utility is given by

$$\frac{\partial u / \partial x_i}{P_i}.$$

This is equal to the marginal utility of income which is obtained when the increase in income can be optimally distributed among all goods, since by (3.3) a marginal increase in expenditures increases the utility by the same amount, whichever good is purchased. Thus

$$\frac{\partial u / \partial x_i}{P_i} = \partial v / \partial I \qquad\qquad i = 1, 2, \ldots, n.$$

(3.4) has the following interpretation. If the price of the *i*-th good is raised by a dollar per unit and consumption of the *i*-th good is fixed, expenditure on that good must increase by x_i dollars, and expenditure on other goods must decrease by the same amount. The utility level would therefore decline by x_i times the marginal utility of income. By (3.3) it does not matter if substitution occurs: at the optimum all goods have the same marginal utility per dollar expenditure.

Combining (3.4) and (3.5) yields *Roy's Identity*:

$$\begin{aligned} x_i &= -(\partial v(p, I) / \partial p_i) / (\partial v(p, I) / \partial I) \\ &\equiv \hat{x}_i(p, I), \qquad\qquad i = 1, 2, \ldots, n, \end{aligned} \qquad (3.6)$$

which is derived in Chapter I without using the Envelope Theorem. $\hat{x}_i(p,I)$ is the uncompensated (or Marshallian) demand function. This result is quite useful: demand functions can be obtained simply by differentiating the indirect utility function.

Next, consider the problem of minimizing the expenditure necessary to achieve a given utility level. In this problem, $p \cdot x$ is minimized under the constraint,

$$u(x) = u, \tag{3.7}$$

for a given u. The minimum expenditure level is a function of prices and the utility level, $E(p,u)$, which is called the *expenditure function.*

If l is the Lagrange multiplier, the Lagrangian is

$$\Phi = p \cdot x + l\,[u - u(x)], \tag{3.8}$$

and

$$p_i = l\,(\partial u / \partial x_i), \quad i = 1,2,\ldots,n. \tag{3.9}$$

By the Envelope Theorem, (2.3),

$$l = \partial E(p,u) / \partial u, \tag{3.10}$$

$$x_i = \partial E(p,u) / \partial p_i \equiv x_i(p,u). \tag{3.11}$$

The latter equation is usually called *Shephard's Lemma* and gives the compensated demand function $x_i(p,u)$.

It can be easily shown that the expenditure function is concave as a function of prices for any fixed utility level. Let p and p' be two arbitrary price vectors and x^* and $x^{*\prime}$ be corresponding optimal consumption vectors. Then

$$E(p,u) = p \quad x^*$$

and

$$E(p',u) = p' \quad x^{*\prime}.$$

Consider a new price vector $\hat{p} = tp + (1-t)\,p'$ for an arbitrary t between 0 and 1, and the corresponding consumption vector $\hat{x}^*$. The following inequalities hold:

$$p \quad x^* \leq p \quad \hat{x}^*,$$

and

$$p' \quad x^* \leq p' \quad \hat{x}^*.$$

Multiplying the first inequality by t and the second by $1-t$ and adding them yields

$$E(tp+(1-t)p',u) = (tp+(1-t)p')\ \hat{x}^*$$
$$\geq tp\ \ x^* + (1-t)p'\ \ x^{*\prime}$$
$$= tE(p,u)+(1-t)E(p',u).$$

Thus $E(p,u)$ is concave with respect to p. If E is twice differentiable, the concavity implies

$$\partial^2 E(p,u)/\partial p_i^2 = \partial x_i(p,u)/\partial p_i \leq 0. \tag{3.12}$$

This shows that price increase for any good does not increase the uncompensated demand for that good, i.e., the own substitution effect is nonpositive. This is used in Equation (I.1.20) of Chapter 1.

Now, we derive the *Slutsky equation*, describing the relationship between the uncompensated and compensated demand functions. For given prices and income, utility maximization yields the indirect utility function $v(p,I)$ and the uncompensated demand function $\hat{x}_i(p,I),\ i=1,2,\ldots,n$. Consider the expenditure minimization given the maximum utility level $u=v(p,I)$. Unless some prices are zero, in which case some technical difficulty appears, the optimal choices coincide and $I=E(p,u)$. The uncompensated demand function therefore satisfies

$$x_i(p,u) = \hat{x}_i(p,E(p,u)), \quad i=1,2,\ldots,n. \tag{3.13}$$

Differentiation of this equation with respect to P_j yields

$$\partial x_i(p,u)/\partial p_j = \partial \hat{x}_i(p,I)/\partial p_j + [\partial \hat{x}_i(p,I)/\partial I][\partial E(p,u)/\partial p_j]$$
$$= \partial \hat{x}_i(p,I)/\partial p_j + [\partial \hat{x}_i(p,I)/\partial I]x_j(p,u), \tag{3.14}$$

where the last term results from substituting according to (3.10). This is the Slutsky equation,

$$\left.\frac{\partial x_i}{\partial P_j}\right|_{u=const.} = \left.\frac{\partial x_i}{\partial P_j}\right|_{I=const.} + x_j\frac{\partial x_i}{\partial I}, \tag{3.15}$$

used in deriving (V.2.27)

Compensated and uncompensated demand functions satisfy another relationship which is also used in deriving (V.2.27). Following an argument similar to that which led to (3.13), we obtain

$$x_i(p,v(p,I)) = \hat{x}_i(p,I), \quad i=1,\ldots,n.$$

Taking a partial derivative with respect to I, we obtain

$$[\partial x_i(p,u)/\partial u][\partial v(p,I)/\partial I] = \partial \hat{x}_i(p,I)/\partial I, \quad i=1,...,n. \tag{3.16}$$

Notes

Discussions in this Appendix owe very much to Dixit (1976). The Envelope Theorem in section 2 was proved by Afriat (1971) and can also be found in Takayama (1974).

References

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