CHAPTER I
THE BASIC MODEL

The simple residential land use model developed in this chapter will be used later to analyze urban externalities. It is helpful, however, to examine competitive equilibrium and optimal allocation in the basic model first, as we do in sections 1 and 2 respectively.

The size and form of a city are at least partially determined by the market decisions of households which buy or rent housing. The decisions involve hundreds of factors such as the size of a lot, the size of a house, distance to the workplace, neighbourhood characteristics, the quality of the schools, the property tax rate and so on. Although all of these factors are important, in this chapter we concentrate on one of the most important: the trade-off between accessibility and lot size. Our households are constantly asking "shall we live in a town-house near work or on a larger lot in the suburbs?".

To avoid unnecessary complications, we make the following assumptions:

(a) In our city *the central business district (CBD) is the only center.* All city residents work in the CBD and commute from the surrounding residential area. This assumption does not, as it turns out, affect the residential pattern: the qualitative results are essentially the same in a multi-centered model.[1]

(b) *All households are identical.* They have the same preferences and the same number of workers. For simplicity, we assume that each household has one worker. All the workers are assumed to have the same skill. These assumptions are important in deriving some of the results. The assumption of the same skill can be easily relaxed, but it is difficult to obtain clear-cut results in a model with different preferences unless the difference in preferences is of a particularly simple nature.

(c) *The only transportation costs incurred are the costs of commuting to the CBD*, either to work or to shop. *The value of commuting time is constant* for any amount of commuting time and the same for all households. Time costs are included in the pecuniary costs of transportation. These assumptions are easily relaxed.[2]

(d) *An individual may reside at only one location.* This assumption eliminates, for example, households with an apartment in the city and a house in the suburbs. The actual number of such households is so small that they can safely be ignored. As will be seen in Appendix I on equality and the

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[1] However, it is not easy to determine the number, locations and sizes of centers. Once they are determined, the residential patterns are obtained in essentially the same way as in a monocentric model.

[2] Henderson (1977), for example, uses a model with time costs.

Benthamite function, this assumption introduces nonconvexity, and is a major departure from the standard neoclassical theory.

(e) *Housing capital can be instanteously adjusted.* Although housing is in reality a durable good, we assume that all the characteristics of houses such as the size of a lot and the size of a house can be changed instantaneously. Ours is, therefore, a city at the imaginary long-run stationary state, in which the capital-land ratio is always perfectly adjusted. Analysis is simplified by this assumption, yet many of the results obtained in the simple polar case carry over to more complex cases. Even if different results are obtained, it serves as a useful reference point and illustrates the basic mechanism. Furthermore, the comparative static results of long-run equilibria suggest the direction of change of an urban economy to policy changes.
If we further assume that the relative prices of housing capital (buildings) and other consumer goods do not change, then by Hicks' Aggregation Theorem houses can be treated as part of the consumer good.[3] The assumption allows us to concentrate on the amount of land used for housing.

(f) *Transportation requires no land input.* We also assume away traffic congestion so that commuting costs are simply a function of the distance from the CBD. This assumption will be relaxed in Chapters IV and V.

(g) *There are no externalities and no public goods.* This assumption will also be relaxed in later chapters. Externalities among producers will be examined in Chapter II; local public goods in Chapter III; traffic congestion in Chapters IV and V; and externalities between different types of individuals in Chapter VI.

1. Market Cities

In this section we analyze competitive equilibrium of a city. The equilibrium spatial structure is examined in subsection 1.1. It is assumed that all residents receive the same income. Because everyone is assumed to have the same utility function, the utility level must be the same everywhere in the city. Land rent, thus, declines with distance from the CBD to offset an increase in commuting costs. As the relative price of land falls, consumption of land increases while consumption of the consumer good decreases. It follows that population density declines with distance from the center, as observed in most cities in the world. Furthermore, if the commuting cost is a linear or concave function of distance, the rent function must be a convex function of distance.

We consider different income classes in subsection 1.2 although we continue to assume that households are identical in all other respects: all households have the same preferences and transportation costs. Under these assumptions, richer households live farther from the center than poorer households if land is a normal good. This result follows from the fact that richer households have a flatter rent curve at the boundary. The rent must fall with distance from the center in order to offset an increase in commuting costs, but the required fall is smaller for richer households since under the normality assumption they consume more land, and therefore benefit more from the same fall in rent.

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[3] See Hicks (1946, pp. 312-313).

In subsections 1.1 and 1.2, the utility levels and the incomes of residents are left undetermined. Two ways of determining these variables are introduced in subsections 1.3 and 1.4. The more popular formulation is that of a *closed city,* which assumes that the population of a city is given. This type of model may be interpreted as dealing with a time period long enough to attain an equilibrium within a city, but too short to allow migration between cities. Since it takes a long time to change the housing stock, this interpretation is somewhat schizophrenic.

It is more consistent to interpret the closed city model as the long-run stationary equilibrium of a closed homogeneous economy with given population, a given number of identical cities and an insignificant rural sector. The population of a single city is then given by simple division.

As a natural extension of this interpretation, we can take the number of cities as a variable. A non-urban sector such as an agricultural sector can also be introduced so that migration between urban and nonurban sectors can be analyzed. These extensions are considered in the next chapter on city sizes.

In subsection 1.4 we examine a *small "open" city,* where openness means that migration of households and transportation of products between cities are costless and otherwise unrestricted. In an open city, commodity prices and the utility level of residents are equal to those in the rest of the economy. When an open city is small compared with the entire economy, any change in allocation within the city will spread over the whole economy and local prices and utility level will not be affected significantly. Prices and the utility level may, therefore, be taken as given for the city.

This model is appropriate when the long-run allocation of a city is the focus. A city administrator, for example, may want to adopt this model to analyze the long-run effects of his policies. The model may also be applied to cities in developing countries with surplus labour, or to cities in a small country which allows free migration.

In both open and closed cities we have to distinguish between the *"absentee-landlord"* case, in which land is owned by absentee landlords who spend their incomes outside the city, and the case of *"public ownership".* In our treatment of public ownership a city government rents the land from agricultural landowners at the agricultural rent and sublets it to households at the market rent, using the net revenue to subsidize city residents equally.

1.1. The Spatial Structure of a Residential City

Consider a city in a featureless agricultural plain. To simplify exposition, we assume that production does not require space, so that the CBD is just a point.[4] The residential zone extends to distance $\bar{x}$ from the CBD. The analysis may be applied to any shape, but it is often easiest to imagine dealing with a circular city. In any ring between radius x and $x + dx$, there are $\theta(x)dx$ units of land available, out of which $L_H(x)dx$ units are used for housing. The structural component of housing is included in the composite consumer good. At the edge of the residential zone the residential

[4] It is not difficult to introduce land use for urban production. See Appendix II for this extension in the context of local public goods.

rent must be equal to the rural rent.

One person from each household commutes to the CBD. The commuting costs, $t(x)$, for a household at a radius *x*, are assumed to be an increasing function of distance from the center:

$$t'(x) > 0 \quad . \tag{1.1}$$

Consumption of the composite consumer good, which includes buildings, and consumption of land for housing are denoted by $z(x)$ and $h(x)$ respectively. Transporting the consumer good is

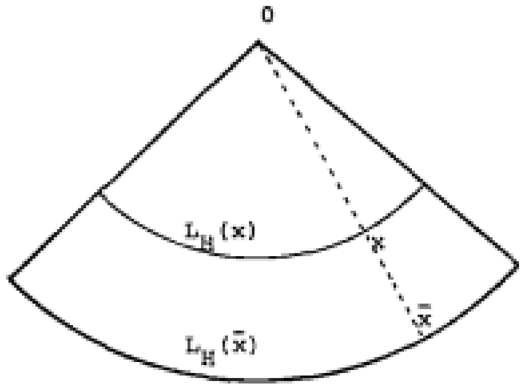

Figure 1. The residential zone

costless. All households have the same quasi-concave utility function,

$$u = u(z, h) \quad . \tag{1.2}$$

We assume that the utility function is appropriately differentiable, although it is not necessary for all the results that follow.

The budget constraint for a household at x is

$$I(x) \equiv y - t(x) = z(x) + R(x)h(x) , \tag{1.3}$$

where $I(x), y,$ and $R(x)$ are respectively the income net of the commuting costs, the gross income, and the residential land rent. The rent function, $R(x)$, provides the rent for a unit area of land at any given radius. The gross income is assumed to be the same for every household. How the income level is determined will be specified later. Note that the consumer good is taken as the numeraire.

A household maximizes the utility function, (1.2), subject to the budget constraint, (1.3). The first order condition for this maximization problem is

$$\frac{u_h}{u_z} = R(x) , \tag{1.4}$$

where subscripts *h* and *z* denote partial derivatives with respect to *h* and *z*. This is the familiar condition that the price ratio and the marginal rate of substitution are equal. From this first order condition and the budget constraint, demands for the consumer good and land can be written as functions of the net income, $I(x) \equiv y - t(x)$, and land rent, $R(x)$:

$$z(x) = \hat{z}(I(x), R(x)) \,, \tag{1.5}$$

$$h(x) = \hat{h}(I(x), R(x)) \,. \tag{1.6}$$

Since these functions describe the levels of demand obtained at a fixed income level, they are nothing but *uncompensated* (*or Marshallian*) *demand functions.* By substituting (1.5) and (1.6) into the utility function, we obtain the *indirect utility function,*

$$v(I(x), R(x)) \equiv u[\hat{z}(I(x), R(x)), \hat{h}(I(x), R(x))] \,, \tag{1.7}$$

which describes the maximum utility level available to consumers, given the net income, $I(x)$, and land rent, $R(x)$.[5]

The demand functions satisfy the following useful relationships obtained by differentiating the budget constraint (1.3):

$$h + R\hat{h}_R + \hat{z}_R = 0 \quad , \tag{1.8}$$

$$R\hat{h}_I + \hat{z}_I = 1 \quad , \tag{1.9}$$

where subscripts *R* and *I* denote respectively partial derivatives with respect to $R(x)$ and $I(x)$. Using these equations, we can see that the indirect utility function satisfies *Roy's Identity*[6]*:*

$$v_R = -v_I h \quad . \tag{1.10}$$

Since households are identical, in equilibrium the utility level must be the same everywhere in the city. Otherwise, households at a place of lower utility level have an incentive to relocate, and the allocation cannot be a market equilibrium. Thus the land

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[5] See Section 3 of Appendix III on the envelope property for discussions of the indirect utility function in conjunction with the Envelope Theorem.

[6] Roy's Identity is derived in the following way. From (1.7), partial derivatives of $v(I, R)$ are given by

$$v_R = u_z\left(\hat{z}_R + \frac{u_h}{u_z}\hat{h}_R\right)$$

$$v_I = u_z\left(\hat{z}_I + \frac{u_h}{u_z}\hat{h}_I\right).$$

In view of (1.8) and (1.9), substitution of (1.4) into these equations yields

$$v_R = -v_I h \quad .$$

See Section 3 of Appendix III for a more elegant way of deriving Roy’s Identity which makes use of the Envelope Theorem.

rent must satisfy

$$v(y-t(x), R(x)) = u = \text{const.} \quad , \tag{1.11}$$

which can be solved for $R(x)$ to yield

$$R(x) = R(y-t(x), u) \quad . \tag{1.12}$$

This function is called the *bid rent function.* It describes the maximum rent which a household can pay at a particular distance from the center if it is to receive the given utility level. If the utility level and the income level are known, the bid rent function gives the equilibrium rent. This is merely a result of the rational behaviour of households. If, for example, the actual rent were lower than the bid rent, it would be possible to achieve a higher utility level, and a rational household would not fail to do so. The actual rent cannot be higher than the bid rent simply because it is impossible to pay any higher rent and achieve the given utility level. The bid rent function is extremely useful in a model with one type (or a few types) of households, since in each type the income and the utility level must be the same at any distance from the center. The bid rent function summarizes, in a single function, the rent profile that is compatible with the given income and utility levels.

At the edge of the city, where $x = \bar{x}$, the residential rent must equal the rural rent R_a:

$$R(\bar{x}) = R_a \quad . \tag{1.13}$$

Given the levels of income and utility, (1.12) and (1.13) completely determine the rent profile. Once the rent profile is determined, the allocation of a city is fully characterized, since (1.5) and (1.6) give the consumption of the consumer good and of land for housing at each location.

In this simple model, the transportation cost function and the utility function completely determine the spatial structure of the city as Figure 2 illustrates. Consider any two locations, x_1, and x_2, where x_1 is closer to the center than x_2. Inspection of the budget constraint (1.3) shows that a budget line intersects the vertical axis at $y-t(x)$. Since the utility level is maximized under the budget constraint, the budget line must be tangent to an indifference curve at the optimum. If the utility level is the same everywhere in the city, households are on the same indifference curve, *u*, at any location *x*. The budget line is thus fully determined and the consumption of the consumer good and land can be read off.

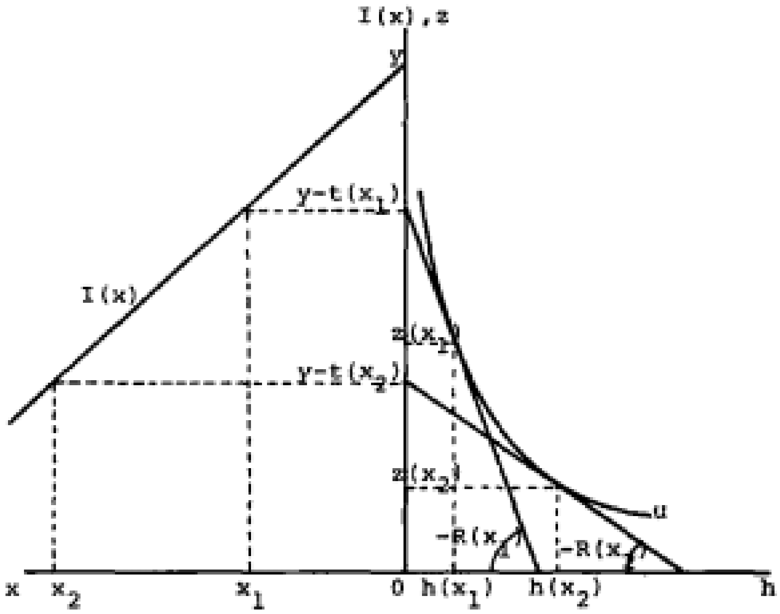

Figure 2. Allocation in the basic model

The bid rent is given by the slope of the budget line. Convexity of indifference curves implied by quasi-concavity of the utility function ensures that the bid rent is lower at x_2 than at x_1. That is, the bid rent curve, $R(I(x),u)$, is a decreasing function of distance x from the center. Furthermore, the lot size increases and the consumption of the consumer good decreases with distance from the center, as households substitute land for the consumer good.

More precise properties can be derived by using calculus. From (1.11) and Roy's Identity (1.10), the rent profile satisfies the following simple relationships:

$$R_I = 1/h(x) \tag{1.14}$$

$$R_u = -1/v_I h(x) \quad . \tag{1.15}$$

Thus, demand for land is a reciprocal of the partial derivative of the bid rent function with respect to income. Differentiating (1.12) and substituting (1.14) yields

$$R'(x) = -t'(x)/h(x) < 0 \, , \tag{1.16}$$

which shows that *the land rent declines with distance from the center.*

If demand functions are obtained for a given utility level instead of a given income level, we have *compensated* (*or Hicksian*) *demand functions:*[7]

$$z(x) = z(R(x),u) \tag{1.17}$$

$$h(x) = h(R(x),u) \tag{1.18}$$

[7] See Section 3 of Appendix III for a derivation of the compensated demand function and its properties from the expenditure function.

The compensated demand functions are useful since the signs of partial derivatives are unambiguous:

$$z_R \geq 0 \tag{1.19}$$

$$h_R \leq 0\,. \tag{1.20}$$

The first inequality is a result of the fact that if there are only two goods, they are always net substitutes. The second inequality represents the elementary property that the (own) substitution effect is negative.

The slopes of $z(x)$ and $h(x)$ are obtained from (1.16), (1.19) and (1.20):

$$z'(x) = z_R R'(x) = -\frac{t'(x)}{h(x)} z_R \leq 0 \tag{1.21}$$

$$h'(x) = h_R R'(x) = -\frac{t'(x)}{h(x)} h_R \geq 0\,. \tag{1.22}$$

The consumption of the consumer good is a nonincreasing function and the lot size a nondecreasing function of distance. The latter property is used by urban economists to explain the fact that the population density declines with distance from the center in most cities.

Differentiating (1.16) again, we obtain

$$R''(x) = -\frac{t''(x)}{h(x)} + \frac{h'(x)}{(h(x))^2} t'(x) \tag{1.23}$$

From (1.22), a sufficient condition for $R''(x) > 0$ is that $t''(x)$ is nonpositive. This yields another well-known result: *if the commuting cost is a linear or concave function of distance from the CBD, the rent function is convex.*

We were able to treat $z(x)$ and $h(x)$ as choice variables because we assumed that housing capital is extremely cooperative. We have ignored a very important aspect of the housing market: the durability of the housing stock. The model therefore describes a long-run stationary state which may never come to exist. In order to introduce durability we would have to develop a dynamic model, making analysis much more complicated.

1.2. Several Income Classes

The above analysis can be easily extended to include different types of households.[8] In this section we consider the case where there are two income classes. For simplicity, and in accordance with empirical observations, land is assumed to be a *normal good:*

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[8] Although everybody is assumed to have the same skill, households can have different incomes since they may own different shares of firms and land.

$$\hat{h}_I[I(x), R(x)] > 0 .$$

Assuming normality, we can show that *there is segregation by income*: the residential zone is divided into two rings, each occupied by one income class. Moreover, we can show that *the richer group lives in a ring farther from the center*, which agrees with the actual residential pattern in most American cities. The argument is quite direct.

Space is occupied by those who are willing to pay the highest rent for it. In other words, the equilibrium rent at any point is simply the highest of the bid rents at that point. Now, the bid rents are functions of income and utility levels, and the rich have higher incomes than the poor: $y^r > y^P$.

At some radius x^*, rich and poor living in the same city must live side by side. This radius is the boundary between two rings of households with different incomes. At this location the two income groups must pay the same rent. From (1.16), the bid rent function is steeper for the lower income group since t' is the same for both groups, and by the normality assumption the lower income group consumes a smaller amount of land. It follows that the richer income group has the higher bid rent outside x^* and lives there. Thus the equilibrium residential pattern is complete segregation with the richer income class living in the outer ring.[9]

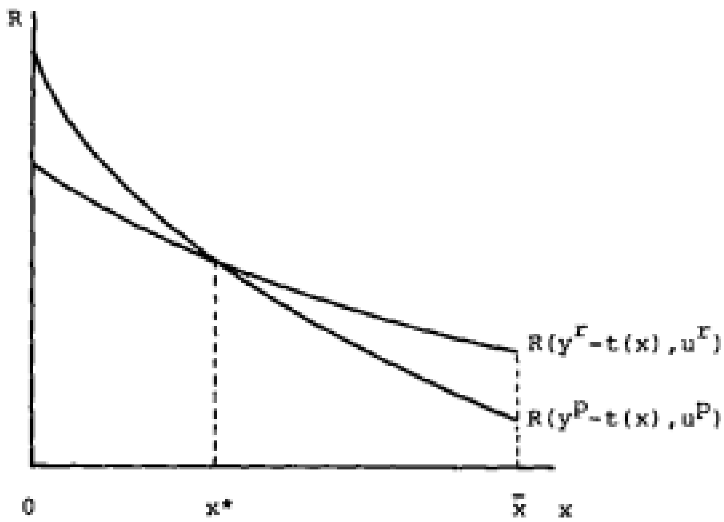

Figure 3. Two income classes

The flatter bid rent curve of the rich can be understood as follows. Suppose that as a poor household moved outwards, the loss of utility due to increased commuting costs was just offset by an increase in utility arising from increased land use. Clearly, this is possible only if the rent on a unit of land falls. But since richer households have larger lot sizes, the same decline in rent allows them larger savings in the total

[9] For arbitrary utility levels, it is possible that the bid rent of one income class is higher than that of the other everywhere in the city. In such a case only one income class lives in the city. The utility levels must be adjusted in order for both groups to live in the city.

expenditures on land. Richer households, therefore, benefit more from the same fall in rent and would be willing to accept a smaller decline in rent as they move outward. The bid rent curve of the rich thus falls less rapidly with distance from the center.

This result has been used to explain the residential pattern observed in the United States. However, it crucially depends on the assumption that all income classes have the same commuting costs. Since time costs constitute a large portion of commuting costs, richer households may live closer to the center if their value of time is much higher than poorer households'. This may explain why the opposite spatial pattern is observed in most cities in Europe, Latin America and Japan, as well as the existence of high-rent luxury apartments near the center of most cities. According to an empirical study (1977) by Wheaton, if time costs are taken into account, the tendency of wealthier households to move to the periphery is weak even in American cities. This suggests that the observed pattern is mainly caused by other factors, such as the concentration of older houses in central cities.

1.3 A Closed City

In the previous subsection, important variables such as incomes and utility levels were left undetermined. In this and the following subsections, different ways of determining them are introduced. For simplicity, we consider cities with only one income class.

The analysis in subsection 1.1 shows that the allocation of a city is completely determined by utility maximization of households and spatial arbitrage, if the utility level, the income level and the size of the city are specified. Since we already have condition (1.13) as one of the three equations required to determine these variables, only two more equations must be specified.

In this subsection, we consider a *closed city;* immigration into and out of the city is impossible and therefore the population is *fixed.* For convenience, the population is identified with the number of households. Denoting the total population of the city by P, the *population constraint* is

$$P = \int_0^{\bar{x}} N(x)dx \tag{1.24}$$

where $N(x)dx$ is the number of households living between x and $x + dx$. Recalling that $L_H(x)$ and $h(x)$ denote respectively the total land available for housing and the lot size at radius x, we can write $N(x)$ as

$$N(x) = \frac{L_H(x)}{h(x)} \tag{1.25}$$

The *aggregate production function* is

$$Y = F(P), \tag{1.26}$$

where all factors other than labor are assumed to be fixed and suppressed. If a city resident is paid the value of the marginal product of labor, the wage rate is given by $w = F'(P)$. If city residents collectively own firms and factors other than labor, a city resident will receive the average product, $F(P)/P$. In either case wages are a fixed amount w if the population is fixed.

Income may differ from wages depending on the treatment of land rent. We consider only the two polar cases; the "*absentee-landlord*" case, and the "*public-ownership*" case. Intermediate cases are left to the reader. In the absentee-landlord case, land is owned by landlords who do not live in the city, and the rent is spent outside the city. The income of a resident is simply the given wage rate:

$$y = w\,. \tag{1.27}$$

(1.24) and (1.27) give our missing two equations and the allocation of the city is completely determined.

The absentee-landlord case is used more often in descriptive analysis to avoid an artificial institutional arrangement. If the optimality of an allocation is a major issue, however, the absentee-landlord case is not convenient because the welfare of absentee landlords has to be taken into account, forcing us to compare utilities of landlords and tenants. We shall therefore adopt the public-ownership framework in normative analysis.

For the public-ownership case we assume the following rather artificial institutional arrangement. The city residents form a government which rents the land for the city from rural landlords. We assume that landlords cannot obtain any monopolistic power, so that the city government needs only to pay the rural rent R_a. The city government, in turn, subleases the land to city residents at the competitively determined rent, $R(x)$. The net revenue is divided equally among households.

There is $\theta(x)dx$ of land between x and $x+dx$, out of which the city sublets $L_H(x)dx$ to city residents and uses the rest for public purposes such as roads and parks. The net revenue of the government is then given by

$$\int_0^{\bar{x}} \left[R(x)L_H(x) - R_a\theta(x)\right]dx \quad .$$

The income of a household is the sum of wages and the "*social dividend"* it receives from the city government:

$$y = w + \frac{1}{P}\int_0^{\bar{x}} \left[R(x)L_H(x) - R_a\theta(x)\right]dx \tag{1.28}$$

We temporarily assume that the entire land is rented to city residents for residential use:

$$L_H(x) = \theta(x) \qquad\qquad 0 \le x \le \bar{x} \tag{1.29}$$

We shall relax this assumption in Chapter IV when we introduce land for transportation use.

(1.28) describes how factor incomes are allocated. If we consider how the goods are allocated, the following constraint is obtained:

$$Pw = \int_0^{\bar{x}} \left\{\left[z(x) + t(x)\right]N(x) + R_a\theta(x)\right\}dx \tag{1.30}$$

The city residents collectively command Pw units of the consumer good, which are consumed or spent on commuting costs and the payment of the rural rent. This constraint is a *resource constraint* that the city faces and will be used in the optimization framework. The equivalence of (1.28) and (1.30) can be readily derived by using the budget constraint (1.3).

1.4 A Small and Open City

A perfectly closed city is one where migration in and out is impossible. It is useful to consider the case in which migration is possible. We assume that migration of households and transportation of products between cities are completely costless. We further assume that the city is so small that any change within the city does not affect the outside world. Prices and the utility level within the city, therefore, equal world levels and may be taken as given.

Since the population size is endogenous in an open city, wages cannot in general be taken as exogenous.[10] Therefore, the income of a household is

$$y = w(P)$$

in the absentee-landlord case, and

$$y = w(P) + \frac{1}{P}\int_0^{\bar{x}} \left[R(x)L_H(x) - R_a\theta(x)\right]dx$$

in the public-ownership case. Either of these equations, if coupled with (1.24), determines the population size and the income level, and thereby completely specifies the resource allocation in the city.

Although it is possible that the city government would be controlled by old residents who treat newcomers differently, as in some of the club theory literature, for example, McGuire (1974), we shall not pursue this line here. We assume that newcomers receive all privileges of citizenship including a share of net city revenue.

If a city is not small but open, a case intermediate between a closed city and a small city is obtained. Given the total population of the economy, the population of the rest of the economy can be expressed in terms of the population, *P*, of the city. When households leave the city, the marginal product of labour rises in the city and falls elsewhere, as a result of diminishing returns. Since migration is free, equilibrium will be reached when the utility level outside the city, $V(P): V'(P) > 0$, equals the utility level in the city:

$$u = V(P).$$

This condition replaces the fixed-population constraint in a closed city and the fixed-utility constraint in a small city. This more general formulation will be used in Chapter VI. Note that the polar cases of $V'(P) = 0$ and $V'(P) = \infty$ yield a small city and a closed city respectively.

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[10] If, however, constant returns to scale are assumed and a resident receives the average product, w is constant. This assumption is quite often made (at least implicitly) in the literature.

2. Optimum Cities

To obtain an optimal allocation, an objective, or criterion, function must be specified. Probably the most natural one is a Benthamite social welfare function which is the sum of the utilities of individual households,

$$\int_0^{\bar{x}} u(z(x), h(x)) N(x) dx . \tag{2.1}$$

Note that the Benthamite social welfare function requires that utility be cardinal.[11] In addition it is commonly assumed that the marginal utility of income decreases as income increases. This is a cardinal property and it is represented by the assumption that the utility function is concave.

We can imagine the Benthamite optimum being achieved as follows.[12] Let an individual choose the optimal resource allocation, including income distribution, based on her own selfish preferences. Decisions must be made, however, "behind the veil of ignorance": she must not know which of the residents she will become. If she has an equal chance of becoming any of the residents, her expected-utility maximization is equivalent to maximizing the Benthamite social welfare function.

It turns out that at the Benthamite optimum the utility level varies with the distance from the center. When land is a normal good, the utility level rises with distance from the CBD. It also turns out that for an appropriate unequal income distribution the corresponding competitive equilibrium exactly replicates the optimum solution.

Theorists have been intrigued to find that the optimal utility levels differ among locations even though the social welfare function is egalitarian. This result is surprisingly robust, at least among additive social welfare functions. It can be explained as follows. Because of the difference in commuting costs, identical households at different locations have different capability to generate utility from the same amount of resource. The Benthamite optimum, therefore, is attained if more resource is allocated to the more efficient households.

As Appendix I shows, the difference in the efficiency with which households realize utility from their commodity bundles arises from the most fundamental properties of our spatial allocation problem. We assumed that a household cannot live at more than one location. Each household, therefore, must choose one location, and every location has an associated commuting cost. Identical households with equal incomes, once they choose different locations and hence different consumption bundles, are in effect no longer identical. If households are able to divide their time among two or more residences, however, every household faces the same opportunity set and the inequality of utility levels will disappear.

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[11] If utility is merely ordinal, any monotonic transformations of a utility function are considered as equivalent. A monotonic transformation can, however, yield a different Benthamite optimum. In order to obtain the same Benthamite optimum, we must assume that utility functions are equivalent only up to linear transformations, i.e., utility is cardinal.

[12] See, for example, Arnott and Riley (1977).

Even if the social welfare function is made more egalitarian by taking a concave transformation of the utility function - that is, if a new social welfare function,

$$\int_0^{\bar{x}} \phi[u(z(x),h(x))]N(x)dx\,,$$

is adopted - the optimal allocation continues to have unequal utility levels. This conclusion follows immediately from the observation that even if we redefine the utility function as $U(\cdot) = \phi(u(\cdot))$, our assumptions on the original utility function still hold for the new one.

The only way of obtaining an equal utility level with an additive social welfare function is to take a limit coinciding with the Rawlsian welfare function, which maximizes the minimum utility level. For example, Dixit (1973) considered the welfare function

$$\int_0^{\bar{x}} -u(z(x),h(x))^{-m}\,N(x)dx$$

and obtained a uniform utility level by taking the limit as $m \to \infty$. Appendix I contains a detailed discussion of why utility levels differ between different locations except in the limit.

Some economists prefer the Benthamite welfare function on the grounds that the Rawlsian welfare function has the undesirable property of ignoring the welfare of all but the poorest individual. Although the Rawlsian function is the only *additive* social welfare function that yields equal utility, there are other nonadditive functions that will do. As shown in Appendix I, equal utility requires social welfare indifference curves to have sufficiently strong kinks on the line where utility levels are equal.

Except in this section we will consider only cases where utility levels are equal for identical households. The reason is twofold. First, this case is mathematically more tractable, and easier to compare with the market equilibrium. Second, readers might object to giving different utility levels to households which differ *only* in the location of their residences.

2.1 A Closed City

In this subsection, we consider optimal allocation of a closed city. Only the public-ownership case is analyzed because in the absentee-landlord case the welfare of absentee landlords must be taken into account, which destroys the simple structure of our problem. The total amount, Y, of the consumer good produced in the city is used for direct consumption, transportation, and the payment of the rural rent. The *resource constraint* for the city is then

$$Y = \int_0^{\bar{x}} [(z(x)+t(x))N(x) + R_a\theta(x)]dx \tag{2.2}$$

which corresponds to (1.30) in the previous section. The city also faces the *population constraint*, (1.24), and the *land constraint*,

$$\theta(x) = N(x)h(x)\,, \quad 0 \le x \le \bar{x} \tag{2.3}$$

The land constraint is obtained by combining (1.25) and (1.29).

The objective function is the Benthamite social welfare function (2.1). The Lagrangian for this problem is

$$\Lambda = \int_0^{\bar{x}} u(z(x),h(x))N(x)dx + \delta\left\{Y - \int_0^{\bar{x}} [(z(x)+t(x))N(x) + R_a\theta(x)]dx\right\}$$
$$+\gamma\left[P - \int_0^{\bar{x}} N(x)dx\right] + \int_0^{\bar{x}} \mu(x)[\theta(x) - N(x)h(x)]dx \,, \tag{2.4}$$

where δ, γ and $\mu(x)$ are respectively Lagrange multipliers associated with (2.2), (1.24) and (2.3). δ can be interpreted as the shadow price of the consumer good, γ the shadow 'price' of a household (with the total production in the city fixed), and $\mu(x)$ the shadow rent of land, all in utility terms. The shadow 'price' of a household may sound peculiar, but it naturally appears in our problem because an increase in population changes the maximum value of the Benthamite social welfare function. The choice variables are $z(x),\ h(x),\ N(x)$, and $\bar{x}$, where $z(x)$, $h(x)$, and $N(x)$ are chosen at each *x* between 0 and $\bar{x}$.

As shown in section 4 of the appendix on optimal control theory, control theory may be applied to this problem and the following first order conditions are immediately obtained:

$$u_Z(z(x),h(x)) = \delta\,, \qquad 0 \le x \le \bar{x}\,, \tag{2.5a}$$

$$u_h(z(x),h(x)) = \mu(x)\,, \qquad 0 \le x \le \bar{x}\,, \tag{2.5b}$$

$$u(x) = \delta[z(x)+t(x)] + \mu(x)h(x) + \gamma\,, \qquad 0 \le x \le \bar{x}\,, \tag{2.5c}$$

$$[u(\bar{x}) - \delta(z(\bar{x})+t(\bar{x})) - \gamma]N(\bar{x}) = \delta R_a\theta(\bar{x})\,. \tag{2.5d}$$

Using (2.5c), (2.5d) can be written

$$\mu(\bar{x}) = \delta R_a \tag{2.5d'}$$

(2.5a) and (2.5b) require that the marginal utility of the consumer good equal its shadow price, and that the marginal utility of land equal the shadow rent at each radius. (2.5c) means that the utility level of a household equals the shadow value of its consumption bundle plus the shadow 'price' of a household. A household at *x* contributes to the social welfare by $u(x)$, but consumes resources whose value is $\delta[z(x)+t(x)] + \mu(x)h(x)$. The difference is the marginal social value of a household, or the shadow 'price' of a household, γ. According to (2.5d'), the shadow rent of the city equals the rural rent times the shadow price of the consumer good at the optimum.

If the utility function is concave and land is a normal good, we can also show that *the utility level rises with distance from the center at the Benthamite optimum.* Differentiating (2.5c) with respect to *x* and substituting (2.5a) and (2.5b) yields

$$\mu'(x) = -\delta t'(x)/h(x) < 0\,. \tag{2.6}$$

Thus the shadow rent is a decreasing function of distance from the center. The desired result follows if the optimal utility level is a decreasing function of the shadow rent.

Implicit differentiation of (1.3) and (1.4) yields the income derivative of the uncompensated demand function for land:

$$\hat{h}_I(I,R) = \frac{u_Z}{D}(u_{hZ}u_Z - u_h u_{ZZ}), \tag{2.7}$$

where

$$D \equiv 2u_{hZ}u_Z u_h - u_h^2 u_{ZZ} - u_Z^2 u_{hh}. \tag{2.8}$$

Since D is nonnegative when the utility function is quasi-concave, (strong) normality of land, $\hat{h}_I > 0$, implies that

$$u_{hz}u_z - u_h u_{zz} > 0. \tag{2.9}$$

From (2.5a) and (2.5b), $z(x)$ and $h(x)$ can be written as functions of $\mu(x)$ and δ: $\tilde{z}(\mu(x),\delta)$ and $\tilde{h}(\mu(x),\delta)$, and the optimal utility level as

$$u^*(x) = u\left[\tilde{z}\mu(x),\delta),\tilde{h}(\mu(x),\delta)\right] \equiv \tilde{u}(\mu(x),\delta).$$

Differentiating (2.5a) and (2.5b), we obtain

$$\frac{\partial \tilde{z}}{\partial \mu} = -\frac{u_{zh}}{u_{zz}u_{hh} - (u_{hz})^2},$$

$$\frac{\partial \tilde{h}}{\partial \mu} = \frac{u_{zz}}{u_{zz}u_{hh} - (u_{hz})^2}.$$

From these equations, we get

$$\frac{\partial \tilde{u}}{\partial \mu} = u_z \frac{dz}{d\mu} + u_h \frac{dh}{d\mu} = \frac{u_h u_{zz} - u_z u_{zh}}{u_{zz}u_{hh} - (u_{zh})^2}. \tag{2.10}$$

This is negative since the denominator is nonnegative when the utility function is concave and the numerator is negative from (2.6). Therefore, from (2.9) we obtain

$$\frac{du^*}{dx} = \frac{\partial \tilde{u}}{\partial \mu}\mu'(x) > 0. \tag{2.11}$$

Thus, the optimal utility level rises with distance from the center.

Next, we examine whether the optimal allocation is attained as a competitive equilibrium. An allocation is a competitive equilibrium in our model if the following conditions are satisfied:

(i) Each household maximizes the utility level with respect to z and h subject to the budget constrain and taking the land rent, $R(x)$, as given.

(ii) No household has an incentive to move to other locations.

(iii) Demand for land equals supply of land.

(iv) Demand for the consumer good equals the supply of the consumer good.

(v) The rent at the edge of the city equals the rural rent.

Defining $R(x) \equiv \mu(x)/\delta$ and $y(x) \equiv (u(x)-\gamma)/\delta$, (2.5a) through (2.5c), (2.5d') and (2.6) can be rewritten

$$u_Z(z(x), h(x)) = \text{const.}, \qquad 0 \le x \le \bar{x},, \tag{2.12a}$$

$$u_h / u_z = R(x), \qquad 0 \le x \le \bar{x}, \tag{2.12b}$$

$$y(x) = z(x) + R(x)h(x) + t(x), \qquad 0 \le x \le \bar{x}, \tag{2.12c}$$

$$R(\bar{x}) = R_a, \tag{2.12d}$$

$$R'(x) = -t'(x)/h(x), \qquad 0 \le x \le \bar{x} \tag{2.12e}$$

Condition (i) is satisfied at the Benthamite optimum since (2.12b) is the first order condition for the problem of maximizing the utility function, $u(z,h)$, subject to the budget constraint, $y(x) = z + R(x)h + t(x)$, with respect to *z* and *h*.

Condition (ii) is satisfied if a household living at any radius *x** achieves its maximum utility at *x**, that is, a household with income $y = y(x^*)$ maximizes the indirect utility function, $v(y - t(x), R(x))$, with respect to *x* at *x**. The first order condition for the maximization is

$$\frac{dv}{dx} = -v_I\left[R'(x)\hat{h}(y - t(x), R(x)) + t'(x)\right] = 0, \tag{2.13}$$

where we used Roy's Identity (1.10), and $\hat{h}(\cdot)$ is the uncompensated demand for land (1.6). (2.12e) ensures that (2.13) is satisfied at the Benthamite optimum. The second order condition is

$$\frac{d^2v}{dx^2} = -v_I\left[t''(x) + R''(x)h(x) + R'(x)(-\hat{h}_I t'(x) + \hat{h}_R R'(x)\right] \le 0. \tag{2.14}$$

Since (2.13) is satisfied at each *x* if $y = y(x)$, we have

$$R'(x)\hat{h}(y(x) - t(x), R(x)) + t'(x) = 0, \qquad 0 \le x \le \bar{x}\ 0. \tag{2.15}$$

Differentiating this equation with respect to *x* yields

$$t''(x) + h(x)R''(x) + R'(x)[\hat{h}_I(y'(x) - t'(x)) + \hat{h}_R R'(x)] = 0. \tag{2.16}$$

Using this equation, the second order condition becomes

$$\frac{d^2v}{dx^2} = v_I \hat{h}_I R'(x) y'(x) \le 0 \qquad 0 \le x \le \bar{x}, \tag{2.17}$$

which is satisfied at the Benthamite optimum since from (2.11) we have

$$y'(x) = u'(x)/\delta > 0 \tag{2.18}$$

if $\hat{h}_I > 0$. This also shows that the income level rises with distance from the center in market equilibrium, and corresponds to the result in subsection 1.2 that if land is normal, richer households live farther away from the center than poorer households.

Conditions (iii), (iv) and (v) are guaranteed by (2.3), (2.2) and (2.12d). Thus the

Benthamite optimum is attained as a competitive equilibrium for a suitable choice of income distribution.

Now we add the constraint that the utility level be equal everywhere in the city, and maximize this equal utility level. Thus, our problem is one of maximizing

$$\int_0^{\bar{x}} uN(x)dx \tag{2.19}$$

subject to the resource constraint (2.2), the population constraint (1.24), the land constraint (2.3), and the constraint that the utility level be equal everywhere in the city,

$$u = u(z(x), h(x)), \qquad 0 \le x \le \bar{x}. \tag{2.20}$$

The Lagrangian for this problem is

$$\begin{aligned}\Lambda &= \int_0^{\bar{x}} uN(x)dx + \int_0^{\bar{x}} \nu(x)[u(z(x), h(x)) - u]dx \\ &+ \delta\left\{Pw - \int_0^{\bar{x}} [(z(x) + t(x))N(x) + R_a\theta(x)]dx\right\} \\ &+ \gamma\left[P - \int_0^{\bar{x}} N(x)dx\right] + \int_0^{\bar{x}} \mu(x)[\theta(x) - N(x)h(x)]dx\end{aligned} \tag{2.21}$$

The only new Lagrange multiplier is $\nu(x)$ which can be interpreted as the weights that have to be attached to the utilities of households at different locations if all households are to obtain equal utility levels.

As shown in section 4 of the appendix on optimal control theory, the first order conditions are

$$\nu(x)u_z - \delta N(x) = 0 \qquad 0 \le x \le \bar{x} \tag{2.22a}$$

$$\nu(x)u_h - \mu(x)N(x) = 0 \qquad 0 \le x \le \bar{x} \tag{2.22b}$$

$$u - \delta[z(x) + t(x)] - \gamma = \mu(x)h(x) \qquad 0 \le x \le \bar{x} \tag{2.22c}$$

$$[u - \delta(z(\bar{x}) + t(\bar{x})) - \gamma]N(\bar{x}) = \delta R_a\theta(\bar{x}) \tag{2.22d}$$

$$\int_0^{\bar{x}} N(x)dx = \int_0^{\bar{x}} \nu(x)dx \tag{2.22e}$$

The difference from the Benthamite case mainly lies in (2.22a). Here, the marginal utility of the consumer good does not need to be equal at different locations, while the utility level is equal. In the Benthamite case, the marginal utility is equal but the utility level is not.

Defining $R(x) \equiv \mu(x)/\delta$ and $y \equiv (u-\gamma)/\delta$, (2.22a) through (2.22e) can be written

$$u_h / u_z = R(x), \qquad 0 \le x \le \bar{x} \tag{2.23a}$$

$$y = z(x) + R(x)h(x) + t(x), \qquad 0 \le x \le \bar{x} \tag{2.23b}$$

$$R(\bar{x}) = R_a \,, \tag{2.23c}$$

$$\frac{1}{\delta}\left[\int_0^{\bar{x}} N(x)dx\right] = \int_0^{\bar{x}} \frac{l}{v_I} Nx)dx \quad . \tag{2.23d}$$

Comparison of these equations with market equilibrium conditions in section 1 shows that the optimal solution exactly coincides with the market allocation of a closed city. (2.23d) shows that the reciprocal of the social value of the numeraire is equal to the average of reciprocals of marginal utilities of income.

2.2 A Small and Open City

In a small, open city it is meaningless to maximize the utility level of city residents because the level is determined independently of the allocation within the city. Under some circumstances, however, maximizing the net product of the city may be of interest: a mining company, for example, building a townsite on its own land would maximize the total product of the city minus the cost of maintaining the utility level required to attract a work force. The profit for such a producer would be

$$F\int_0^{\bar{x}} N(x)dx - \int_0^{\bar{x}} [(z(x)+t(x))N(x) + R_a\theta(x)]dx \tag{2.24}$$

Labour costs do not include the land rent that workers pay since it is paid to the company.

The net product (2.24) is maximized under the land constraint (2.3) and the utility constraint,

$$u(z(x), h(x)) = \bar{u} \qquad 0 \le x \le \bar{x}. \tag{2.25}$$

where $\bar{u}$ is the exogenously given utility level. Note that since the population of the city is a choice variable in an open city, the population constraint can be ignored.

The Lagrangian for this problem is

$$\begin{aligned}\Lambda = F[\int_0^{\bar{x}} N(x)dx] - \int_0^{\bar{x}} [(z(x)+t(x))N(x) + R_a\theta(x)]dx \\ + \int_0^{\bar{x}} \nu(x)[u(z(x), h(x) - \bar{u}]N(x)dx \\ + \int_0^{\bar{x}} R(x)[\theta(x) - N(x)h(x)]dx \,.\end{aligned} \tag{2.26}$$

The first order conditions become, after simple manipulations,

$$u_h/u_z = R(x)\,, \qquad 0 \le x \le \bar{x}\,, \tag{2.27a}$$

$$F' = z(x) + R(x)h(x) + t(x)\,, \qquad 0 \le x \le \bar{x}\,. \tag{2.27b}$$

Considering $R(x)$ as land rent, we can observe that these optimality conditions coincide exactly with the market equilibrium conditions of the absentee-landlord case of the open city if workers earn wages equal to the value of marginal productivity of labor.

Thus the market equilibrium is the optimal solution in this case as well.

Notes

The theory of residential land use which is described in this chapter was first established by Alonso (1964) following the pioneering work of Wingo (1961). Many urban economists have extended Alonso's framework. Extensive empirical research has also been carried out. These efforts have culminated in Muth (1969) and Mills (1972a,b).

The indirect utility function approach adopted in this chapter was introduced into an urban residential land use model by Solow (1973). This approach has proved to be very useful in deriving qualitative results.

The single-center assumption was relaxed by Romanes (1976) and White (1976). In a two dimensional case, introduction of subcenters gives rise to complicated partial differential equations which are very difficult to analyze.

More than one income class was introduced by Beckman (1969) and Solow (1973) among others. Beckman considered the case of Pareto income distribution. Beckman's solution was not correct since, as pointed out by Montesano (1972), he ignored boundary conditions (among other things). Our treatment of different income classes is based on Solow's. Miyao (1975) analyzed the dynamic stability of boundaries between different income classes. Empirical research on spatial residential patterns with several income classes was carried out by Wheaton (1977).

Time costs of commuting were included in Alonso's original formulation, though later studies tend to ignore time costs by considering the pecuniary cost as a surrogate. As discussed in subsection 1.2, the inclusion of time costs tends to weaken the tendency of the richer households to live farther from the center since the rich's value of time is higher than the poor's, making commuting costs for the rich greater than for the poor.

Models with durable housing stock were analyzed by Fujita (1976a,b), and Anas (1976). Since dynamic aspects must be taken into account in this case, the analysis becomes much more complicated.

Definitions of closed and open cities were introduced by Wheaton (1974) in his comparative static analysis.

The Benthamite optimal city was first analyzed by Mirrlees (1972). He discovered that utility levels are not equal at the Benthamite optimum. Riley (1973), (1974) further analyzed this property using different social welfare functions. The product of individual utilities was used in Riley (1973) as the social welfare function, and a general class of concave and additive social welfare functions in Riley (1974). He derived a result parallel to ours: when land is a normal good and when there is no preference for location *per se*, individuals further out will receive greater utility levels at the optimum. Our illustration in Appendix I of the reason why unequal utility levels are obtained at the optimum is largely based on Arnott and Riley (1977) and Levhari, Oron and Pines (1978).

The Rawlsian case was considered by Dixit (1973). The method of maximizing the utility level under the constraint that the utility level be equal everywhere in the city was adopted by Oron, Pines and Sheshinski (1973).

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