CHAPTER IV

TRAFFIC CONGESTION AND LAND USE FOR TRANSPORTATION: OPTIMUM AND MARKET CITIES

Traffic congestion probably induces the most important kind of externality in cities; the waste of resources and time in inefficient transportation may be enormous. In this chapter we extend the basic model to examine commuter congestion when city land is used for transportation. A new problem we face is how to allocate land between transportation and residential uses, or how much road to build.

Traffic produces a variety of externalities, noise, pollution, and risk of injury among them, which affect people whether they are traveling or not. Traffic congestion, however, tends to affect travelers most. Each additional vehicle on the road adds to the congestion and increases the travel time for others. Since the additional traveler decides to travel on the basis of her own costs, and does not have to compensate other travelers for the increased costs she imposes on them, her decision may be socially inefficient.

Decisions may be inefficient in various ways: it may be better to use less congested roads, or other modes of transportation, to travel at a less congested time, or less frequently, and so on. In this chapter, we concentrate on the distortion of residential decisions, assuming that no other decisions can be changed. Households are not charged for costs they impose on others, so they pay less than the social cost for their transportation. Since land rents reflect the differential commuting costs, they are also distorted; and the lot sizes chosen by households are therefore socially inefficient.

The obvious way of achieving an efficient allocation is to levy congestion tolls equal to the costs that a commuter imposes on others. In practice, however, it is technically and politically difficult to introduce congestion tolls.

Allocation of land between transportation and residential uses introduces another complication. Policy makers generally presume that market prices correctly reflect the social marginal value of goods. This presumption, however, is erroneous if congestion tolls are not levied. The private transportation costs are different from the social transportation costs, and market rents are not equal to social rents. The usual benefit-cost criterions based on market prices are, therefore, misleading.

In the next chapter we analyze the "second best" solution under which congestion tolls are not levied but social benefits and costs rather than market benefits and costs are used as a criterion for building roads. In this chapter, however, we compare the market allocation with the optimum allocation based on true social costs. At the optimum, congestion tolls are levied and the amount of land allocated to roads is optimal, while at the market allocation congestion tolls are not levied and roads are built according to the erroneous benefit-cost criterion based on market prices.

For the sake of simplicity we make the following (drastic) simplifying assumptions about the transportation sector.

(a) Automobiles are the only mode of transportation. Although extending the model to include alternate modes would introduce many interesting problems, the analysis would require at least another chapter.

(b) Circumferential travels are costless. This assumption allows us to maintain the one-dimensional framework. We may imagine that there are so many radial roads that any household can reach one of them with negligible costs. Miyao (1978) relaxed this assumption and considered a two-dimensional rectangular city. In order to carry out this extension, he had to simplify other aspects of the model. Although the two-dimensional case introduces the problem of route choice, we do not expect qualitatively different results.

(c) All commuters arrive at and leave the CBD at the same time, and travel at the same speed. This assumption simplifies the analysis greatly since the traffic volume at each radius can be represented by the number of workers living outside the radius. In reality, people are probably brighter than this, and try to avoid the peak time. Our assumption essentially describes the upper limit of urban congestion.

(d) There are no road construction costs. The only costs of building roads are the opportunity cost of land. This assumption can be easily relaxed.

(e) Allocation within the CBD (central business district) can be ignored. In effect, we assume that commuting costs inside the CBD are zero. This assumption was relaxed by Livesey (1973) and Sheshinski (1973) in a model simpler than ours.

(f) Time costs can be ignored. This is consistent with previous chapters, and does not affect the results.

1. The Model

Two new elements are required to extend the transportation sector of the basic model in Chapter I. First, transportation is assumed to require land. Land allocated to transportation use at radius *x* is denoted by $L_T(x)$. The land constraint becomes

$$L_H(x) + L_T(x) \le \theta(x), \tag{1.1}$$

where, as in Chapter I, $L_H(x)$ denotes the amount of residential land.

We continue to ignore the allocation within the CBD and assume that the residential zone stretches from $x=0$ to $x=\bar{x}$. We do not, however, assume that the CBD is a point. This change is made because, if $\theta(0)$ is zero or close to zero, all the available land is devoted to roads near the CBD. In such a case, the nonnegativity constraint, $L_H(x) \ge 0$, for $0 \le x \le \bar{x}$, is binding, and we obtain a corner solution. For simplicity, we assume that enough land is available near the center to preclude the corner solution.

The second new element is traffic congestion: commuting costs for each individual depend on the number of others using the same road at the same time. Specifically, it is assumed that the commuting cost per mile per household at radius x is a function of the volume of traffic *T*(*x*) and the amount of land allocated for transportation $L_T(x)$ at that radius:

$$g(T(x), L_T(x)), \tag{1.2}$$

where the cost increases as the volume increases,

$$g_T(T, L_T) > 0\,, \tag{1.3}$$

and decreases as more land is used as roads,

$$g_L(T, L_T) < 0\,. \tag{1.4}$$

We concentrate on the *total* amount of land used for roads at each radius, and do not analyze how the width of an individual road is determined. In this chapter the width of the road refers to the total amount of land used for transportation at a radius, instead of the width of an individual road.

Commuting costs incurred by a household living at x are

$$t(x) = \int_0^x g(T(x'), L_T(x'))dx'\,. \tag{1.5}$$

Differentiation of this equation with respect to x yields the following differential equation:

$$t'(x) = g(T(x), L_T(x))\,. \tag{1.6}$$

This differential equation, with the boundary condition at $x = 0$,

$$t(0) = 0 \quad , \tag{1.7}$$

is equivalent to (1.5).

Since all commuters arrive at and leave the CBD at the same time, and that they travel at the same speed, the traffic volume at a radius x is equal to the number of households living outside *x*:

$$T(x) = \int_x^{\bar{x}} N(x')dx'\,, \tag{1.8}$$

where *N(x)dx* is the number of households living between *x* and *x+dx*, and is given by (I.1.25): $N(x) = L_H(x)/h(x)$. This is equivalent to the differential equation,

$$T'(x) = -L_H(x)/h(x)\,, \tag{1.9}$$

with the boundary condition,

$$T(\bar{x}) = 0\,. \tag{1.10}$$

t(*x*) is usually called the *private transportation cost* and is different from the *social transportation cost*, since it does not include the external costs imposed on other commuters. The social transportation cost, $G(T, L_T)$, at radius x is an increase in the total transportation cost there, $Tg(T, L_T)$, caused by a marginal increase in traffic:

$$G(T, L_T) \equiv \partial\left[Tg(T, L_T)\right]/\partial T$$

$$= g(T, L_T) + Tg_T(T, L_T)\,. \tag{1.11}$$

An additional car on the road causes more congestion and increases the transportation costs of other commuters by g_T. Since there are *T* cars on the road, the total increase in the costs for other travelers is Tg_T. This external cost must be added to the private transportation cost, g. In the transportation economics literature, the private transportation cost is sometimes called the *average transportation cost*, and the social transportation cost is called the *marginal transportation cost* for the obvious reason.

2. A Closed City

In a closed city the population of the city, P, is given. The population constraint (I.1.24) gives the boundary condition for (1.9) at $x = 0$:

$$T(0) = P. \tag{2.1}$$

To save space we analyze only the public-ownership case, which is slightly simpler than the absentee-landlord case. In our version of public ownership the resource constraint is

$$\int_0^{\bar{x}} [zL_H / h + Tg(T, L_T) + R_a\theta] dx \le Pw. \tag{2.2}$$

The available resource, Pw, is spent on the consumer good, zL_H / h, commuting costs, Tg, and the rural rent, $R_a\theta$. This constraint is different from the constraint (I.1.30) in Chapter I in the following two respects. First, equality is replaced by inequality. This does not change the conclusions because the constraint holds with equality both at the optimum and in market equilibrium. For technical reasons, the inequality constraint is more convenient in this and the next chapters, since the associated Lagrange multiplier can be signed. Second, the transportation cost, tN, is replaced by Tg. Equivalence of these two formulations can be easily seen by integration by parts.

2.1. The Optimum City

In the optimum city the sum of utilities,

$$\int_0^{\bar{x}} [uL_H(x) / h(x)] dx, \tag{2.3}$$

is maximized under the constraints (1.1), (1.9), (1.10), (2.1), (2.2), and the equal-utility constraint,

$$u(z(x), h(x)) = u, \qquad 0 \le x \le \bar{x}. \tag{2.4}$$

The control variables are the consumptions of the consumer good and housing, $z(x)$ and $h(x)$, and the total widths of the road and the residential area, $L_T(x)$ and $L_H(x)$; and the control parameters are the utility level, u, and the physical city size, $\bar{x}$.

The Theorem of Hestenes, which is stated in the appendix on optimal control theory, can be applied to this problem. The Hamiltonian is

$$\phi = [u - \lambda(x)]L_H(x)/h(x) - \delta[z(x)L_H(x)/h(x) + T(x)g(T(x), L_T(x)) + R_a\theta(x)], \tag{2.5}$$

where $\lambda(x)$ and δ are adjoint variables associated with the differential equation (1.9) and the isoperimetric constraint (2.2) respectively. δ is a constant, and satisfies

$$\delta\left\{Pw-\int_0^{\bar{x}}\left[zL_H/h+Tg+R_a\theta\right]dx\right\}=0, \qquad \delta\geq 0. \tag{2.6}$$

As in previous chapters, δ can be interpreted as the shadow price of the consumer good in utility terms. The ad joint variable, $\lambda(x)$, satisfies the adjoint equation

$$\frac{\partial\phi}{\partial T}=-\lambda'(x)=-\delta\left[g(T,L_T)+Tg_T(T,L_T)\right] \tag{2.7}$$

The second equality shows that $\lambda'(x)$ equals the shadow price of the consumer good times the social transportation cost at *x* defined in section 1. Thus $\lambda'(x)$ is the social cost of transportation in utility terms. The first equality confirms this interpretation, since it says that a marginal increase in traffic at radius *x* decreases the sum of utilities by $\lambda'(x)$.

According to the maximum principle, the Hamiltonian must be maximized under the constraints (1.1) and (2.4). The Lagrangian for this problem is

$$\psi=\phi+\nu(x)\left[u(z(x),h(x))-u\right]+\mu(x)\left[\theta(x)-L_H(x)-L_T(x)\right], \tag{2.8}$$

where $\nu(x)$ and $\mu(x)$ are respectively the Lagrange multipliers for the constraints (2.4) and (1.1). The first order conditions

are

$$\frac{\partial\psi}{\partial L_H}=\frac{u-\lambda(x)-\delta z(x)}{h(x)}-\mu(x)=0 \tag{2.9a}$$

$$\frac{\partial\psi}{\partial L_T}=-\delta Tg_L(T,L_T)-\mu(x)=0 \tag{2.9b}$$

$$\frac{\partial\psi}{\partial h}=-\frac{u-\lambda(x)-\delta z(x)}{h(x)}N(x)-\nu(x)u_h=0 \tag{2.9c}$$

$$\frac{\partial\psi}{\partial z}=-\nu(x)u_z-\delta N(x)=0 \quad . \tag{2.9d}$$

$\mu(x)$ must satisfy the condition that

$$\mu(x)\left[\theta(x)-L_H(x)-L_T(x)\right]=0, \qquad \mu(x)\geq 0, \tag{2.10}$$

and can be interpreted as the shadow rent of land in utility terms.

The transversality condition for $\bar{x}$ is

$$\psi(\bar{x})=\left[u-\lambda(\bar{x})-\delta z(\bar{x})\right]L_H(\bar{x})/h(\bar{x})-\delta\left[T(\bar{x})g(T(\bar{x}),L_T(\bar{x}))+R_a\theta(\bar{x})\right]=0, \tag{2.11}$$

which simply says that the city should not extend beyond the point where the marginal social contribution of developing an additional unit of land is zero. The transversality condition for *u*,

$$\int_0^{\bar{x}} N(x)dx = \int_0^{\bar{x}} \nu(x)dx \qquad (2.12)$$

is the same as (I.2.22e) and has the same interpretation.

To simplify the interpretations of the optimality conditions, it is convenient to recast shadow prices in terms of the consumer good. Define

$$\tau(x) \equiv \frac{1}{\delta}[\lambda(x) - \lambda(0)]. \qquad (2.13)$$

Then $\tau(x)$ satisfies both

$$\tau(0) = 0 \qquad (2.14)$$

and

$$\tau'(x) = g(T, L_T) + Tg_T(T, L_T), \qquad (2.15)$$

and can be interpreted as the social transportation cost of commuting from radius x to the center. Similarly, the social rent at x is

$$R(x) \equiv \mu(x)/\delta. \qquad (2.16)$$

Equation (2.9a) may be interpreted as the optimal household budget. We can rewrite (2.9a) as

$$u = \delta z(x) + \mu(x)h(x) + \lambda(x).$$

Dividing through by δ, defining

$$y \equiv [u - \lambda(0)]/\delta, \qquad (2.17)$$

and using (2.13) and (2.16), we obtain

$$y = z(x) + R(x)h(x) + \tau(x). \qquad (2.18)$$

This equation expresses the socially optimal allocation of household income at x if y is the income, $R(x)$ the market rent, and $\tau(x)$ the commuting costs. Then, by (2.15), households must pay the social transportation costs, or the private transportation costs plus the costs of externalities imposed on other travelers. In other words, some way must be found to collect a congestion toll if the price system is to achieve the optimum city.

Notice that in this simple model congestion tolls can be levied according to the location of residence. A household living at x should pay the amount

$$\int_0^x T(x')g_T(T(x'), L_T(x'))dx',$$

of congestion tolls. However, this kind of distance tax is not optimal in a more general model in which households can choose when to travel or the best mode among several modes of transportation.

Rewriting Equation (2.9b) as we did (2.9a), we obtain

$$-Tg_L(T, L_T) = R(x). \qquad (2.19)$$

Now since $-Tg_L(T, L_T)$ is simply the marginal reduction in total transportation costs from widening the road at x with the traffic volume fixed, (2.19) reveals that at the optimum the marginal reduction in transportation costs from widening the road equals the shadow rent. For later use, we define $-Tg_L(T, L_T)$ as the market benefit, *B*(*x*), of widening the road:

$$B(x) \equiv -T(x) g_L(T(x), L_T(x)) . \tag{2.20}$$

Combining (2.9c) and (2.9d), and solving for u_h / u_z yields

$$\frac{u_h}{u_z} = R(x) , \tag{2.21}$$

which says that the marginal rate of substitution between land and the consumer good equals the shadow rent at the optimum. This condition is obtained if a household maximizes utility and pays the congestion tax, and therefore allocates its budget according to (2.18).

The transversality condition (2.11) becomes[1]

$$R(\bar{x}) = R_{a'} \tag{2.22}$$

that is, the urban rent at the edge of the city equals the rural rent.

Thus the optimum solution can be attained by the decentralized market mechanism if three conditions are met: all households are given the equal income y; congestion tolls equal to the external costs, Tg_T, are levied at each *x*; and roads are built according to the benefit-cost criterion, equating the marginal reduction of transportation costs from widening the road to the market rent.

Note that the marginal benefits of the road are given by the marginal reduction of transportation costs with the volume of traffic fixed. This is true even though the construction of a new road changes the allocation of the entire economy: the change in commuting costs induces a change in land rent, and hence in the consumption decisions of households, which changes the residential structure of the city. Due to the envelope property, all the indirect effects cancel out each other and the benefits are simply given by the direct saving in transportation cost.[2] This is a general property of the first best optimum. As will be shown in the next chapter, however, the effects of induced changes do not wash out in the second best world where congestion tolls are not allowed.

When we consider the relationship between the total congestion tolls and the total land rent of the road, one of the standard results from production theory is obtained: profit is negative under marginal cost pricing when there are increasing returns to scale, zero in the constant returns case, and positive in the decreasing returns case. First, consider the case where transportation technology exhibits constant returns to scale: the average transportation cost, *g*, remains the same if the volume of traffic, *T*, and the width of the road, L_T, are increased with the same proportion. In the constant-returns-to-scale case, $g(T, L_T)$ can

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[1] We have been able to prove this only in the case where $g(T, L_T)$ is finite at $\bar{x}$ and $g_L(0, L_T) < \infty$ for all $L_T > 0$. In that case $T(\bar{x}) g(T(\bar{x}), L_T(\bar{x})) = 0$, and $L_H(\bar{x}) = \theta(\bar{x})$, and hence (2.11) implies $R(\bar{x}) = R_a$. The first equality is obvious from $T(\bar{x}) = 0$. The second equality is obtained since otherwise $R(\bar{x})$ is zero from (2.19) and $T(\bar{x}) = 0$. From (2.11) and (2.9a), this implies that $R_a \theta(\bar{x}) = 0$, which cannot happen if $R_a > 0$ and $\theta(\bar{x}) = 0$.

[2] Wheaton (1977) and Arnott (1976) observed this well-known result in the context of urban land use.

be written as

$$g(T, L_T) = \tilde{g}(T / L_T).$$

Then, the total congestion tolls at *x* are

$$T^2 g_T = (T(x)^2 / L_T(x))\tilde{g}, (T(x) / L_T(x)),$$

and the total land rent at *x* is

$$-TL_T g_L = (T(x)^2 / L_T(x))\tilde{g}'(T(x) / L_T(x)).$$

Thus the congestion tolls exactly cover the land rent of the road at each radius.

A proportionate increase of *T* and L_T decreases the average transportation cost, *g*, in the case of increasing returns to scale, and increases in the case of decreasing returns to scale. Therefore, for a change in *T* and L_T satisfying

$$\frac{dT}{T} = \frac{dL_T}{L_T},$$

the corresponding change in *g*,

$$\begin{aligned} dg &= g_T dT + g_L dL_T \\ &= (Tg_T + L_T g_L) L_T dL_T, \end{aligned}$$

is negative in the case of increasing returns and positive in the case of decreasing returns. Thus the total congestion tolls are less than the total land rent in the increasing returns case,

$$T^2 g_T < -TL_T g_L,$$

and greater in the decreasing returns case,

$$T^2 g_T > -TL_T g_L.$$

If the transportation authority pays for land rent of the road and collects congestion tolls, its budget is balanced in the constant returns case, it makes a profit in the decreasing returns case, and suffers a loss in the increasing returns case, which is analogous to the results in the usual production theory.

2.2. The Market City

Let us consider an allocation where the congestion tolls cannot be levied. The width of the road is not determined by the market, but by the benefit-cost criterion based on market prices (to be explained below).

When congestion tolls are not levied, households pay only the private transportation costs, *t*(*x*), given by (1.6) and (1.7). Assuming that all households receive the same income *y*, we obtain the budget constraint (I.1.3),

$$y = z(x) + R(x)h(x) + t(x) \qquad 0 \le x \le \bar{x},$$

where $R(x)$ is land rent. The first order condition for utility maximization is given by (I.1.4):

$$\frac{u_h}{u_Z} = R(x).$$

The utility level must be the same everywhere in the city because of spatial arbitrage. This is equivalent to the condition

$$h(x)R'(x) + t'(x) = 0, \qquad 0 \le x \le \bar{x}, \tag{2.23}$$

which is obtained from (I.1.16).

Since the residential rent is equal to the rural rent at the edge of the city, we have (I.1.13):

$$R(\bar{x}) = R_a.$$

In the public-ownership case on which we shall concentrate, the differential rent is returned to residents. Thus the income level is given by

$$y = w + \frac{1}{P}\{\int_0^{\bar{x}} R(x)L_H(x)dx - \int_0^{\bar{x}} R_a\theta(x)dx\}. \tag{2.24}$$

It is easy to see that this is equivalent to (2.2) with equality.

It is assumed that the (erroneous) benefit-cost criterion based on market prices is adopted to determine the allocation of land between housing and transportation uses. Roads are widened until the market benefit equals the market rent. The market benefit, $B(x)$, is the reduction of transportation costs from a marginal increase in land used for roads, which is given by (2.20). Then we have

$$-T(x)g_L(T(x), L_T(x)) = R(x) \tag{2.25}$$

in equilibrium. Note that this is the same as the benefit-cost criterion (2.19) adopted in the optimum city. Although this naive benefit-cost criterion leads to the optimum allocation of land when congestion tolls are levied, it is no longer optimal in the absence of congestion tolls.

Since no available land is left vacant unless the rent is zero, (1.1) holds with equality:

$$L_H(x) + L_T(x) = \theta(x), \qquad 0 \le x \le \bar{x}. \tag{2.26}$$

Comparing these equations with those obtained in the optimum city, we can see that the only difference lies in transportation costs. In the market city residents pay the private (or average) transportation cost, while in the optimum city they also pay congestion tolls, which make up the difference between the private and social (or marginal) transportation cost.

2.3. Comparison Between the Optimum and Market Cities

In this section the optimum and market cities characterized in the previous sections are compared. Unfortunately, the complexity of the model prevents us from carrying out the comparison in the general case. We, therefore, calculate numerical examples using the Cobb-Douglas type utility function.

$$u = h^{\alpha} z^{1-\alpha}, \tag{2.27}$$

and the Vickrey type transportation cost function without a constant term,

$$t'(x) = g(T/L_T)^{K} , \tag{2.28}$$

where g and k are positive constants.

These functions are chosen for the convenience of computation and are not quite realistic. The properties of the functions are as follows. The Cobb-Douglas utility function (2.27) implies that the proportion of income net of commuting costs spent on land is always α. In other words, the income elasticity of demand for land is one and the price elasticity is minus one. The transportation cost function (2.28) represents constant returns to scale in transportation technology. Since there is no constant term, transportation costs are zero when there is no other car on the road. Transportation costs rise when the traffic density, T/L_T, or the number of travelers per unit width of the road, rises. The elasticity of transportation costs with respect to traffic density is *k* and constant.

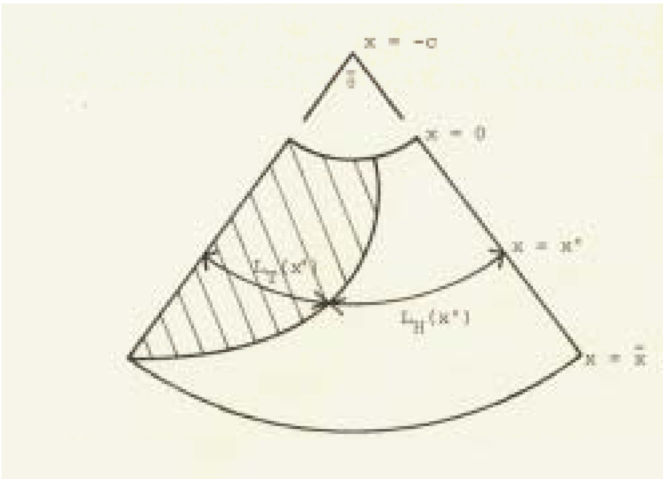

Figure 1. The spatial configuration of the city

The city is assumed to be circular, although not necessarily a complete circle. Since commuting costs in the CBD are zero by assumption (e), we need only consider the residential zone, where the supply of land is

$$\theta(x) = \bar{\theta}(x + c), \tag{2.29}$$

with positive constants $\bar{\theta}$ and *c*. The constant *c* is chosen so that roads do not cover all the land at $x = 0$. In the numerical calculations, $\bar{\theta} = 2$ and $c = 50$.

The results of calculations are shown on Tables 1 and 2.[3] In Table 1, *k* is assumed to be 1 and *g* to be 10^{-5}. α is assumed to be 0.2, which means that a fifth of the income net of

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[3] For the details of calculations, see the Appendix to Chapter V, Part I of Kanemoto (1977).

transportation costs is spent on land. It should be remembered that actual housing is included in the consumer good. The number of households in the city is 100,000, and 1 unit of resources expressed in terms of the consumer good is available for each household. The rural rent is 1 per unit of land.

Table 1

Comparison between Optimum and Market Cities: $k = 1$

	Optimum	Market
Rent at radius *0* $(R(0))$	31.3	14.9
Income per household (y)	1.30	1.03
City size $(\bar{x})$	94.29	120.3
Utility level (u)	0.3955	0.3640
Total area $(\times 10^3)$	6.39	12.0
Total area of roads $(\times 10^3)$	2.14	5.88
Total rent $(\times 10^4)$	1.91	1.50
Total transport costs $(\times 10^4)$	1.72	2.79
$g = 10^{-5}, k = 1.0, w = 1, P = 100{,}000, R_a = 1.0$ $\alpha = 0.2, \theta(x) = 2(x + 50)$		

Table 2

Comparison between Optimum and Market Cities: $k = 2$

	Optimum	Market
Rent at radius *0* $(R(0))$	18.9	5.22
Income per household (y)	1.33	0.83
City size $(\bar{x})$	123.0	177.7
Utility level (u)	0.4450	0.3619
Total area $(\times 10^4)$	1.26	2.91
Total area of roads $(\times 10^3)$	5.10	18.5
Total rent $(\times 10^4)$	1.85	1.22
Total transport costs $(\times 10^4)$	1.34	2.23
$g = 0.5 \times 10^{-8}, k = 2.0, w = 1, P = 100{,}000, R_a = 1.0$ $\alpha = 0.2, \theta(x) = 2(x + 50)$		

There is a striking difference in physical city size between the optimum and market cities: the length of the residential zone of the optimum city is just over three quarters of that of the market city, and the total area of the residential zone (including the road) is just over a half. Because congestion tolls are levied in the optimum city, the land rent tends to be higher and consequently the optimum city is denser than the market city.

The rent at $x = 0$ in the optimum city is more than twice as high as that in the market city, and the total land rent of the residential land in the optimum city is greater than that of the market city even though the market city is considerably bigger. The total rent is 19.1% of the total available resources in the optimum city and 15% in the market city.

In the optimum city the total transportation cost not including congestion tolls is about 62% of those in the market city. Transportation costs constitute 17.2% of the total available resources in the optimum city and 27.9% in the market city. Thus the absence of congestion tolls results in the excessive use of resources in transportation. Since $k = 1$, congestion tolls in the optimum city equal the private transportation cost. This means that when congestion tolls are included, the total commuting costs paid by households are twice as much as the total transportation costs calculated in Table 1. Therefore, although less resources are devoted to transportation in the optimum city than in the market city, households pay more commuting costs in the optimum city if we include congestion tolls. Of course, the revenue from congestion tolls is returned to the city residents in our model, and congestion tolls do not represent any consumption of resources.

The total land allocated to housing is greater in the market city. On the average, therefore, residents in the optimum city consume less land. Notice, however, that housing consumption need not decrease because it is a part of the composite consumer good. Since the total transportation costs (excluding congestion tolls) are smaller in the optimum city, the total consumption of the consumer good is greater. This overwhelms the decrease of the consumption of land and the utility level is higher in the optimum city. Thus the main advantage of the optimum city lies in the fact that the total transportation costs are reduced through dense habitation.

Notice that household income y is 1.3 although we assumed that only one unit of the consumer good was available to each household. The difference is the average expenditure on rent and congestion tolls which is returned to city residents in the public-ownership case.

The road width functions are plotted in Figure 2. The superscripts ^{0} and ^{m} denote respectively the optimum and market solutions. The road in the market city is wider than that

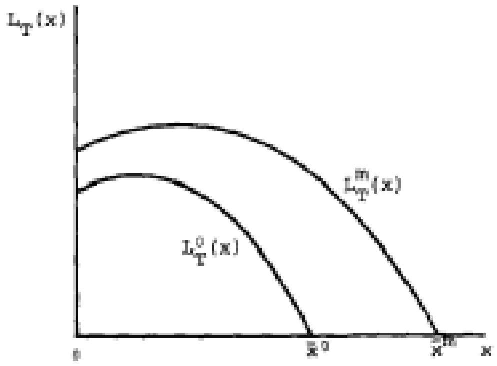

Figure 2
Optimum and Market Road Width Functions: A Closed City

in the optimum city everywhere in the city. In this sense, the benefit-cost criterion based on market prices has a tendency to overinvest in roads. The ratio between the width of the road and the available land is plotted in Figure 3. In both optimum and market cities the ratio decreases monotonically with distance from the center.

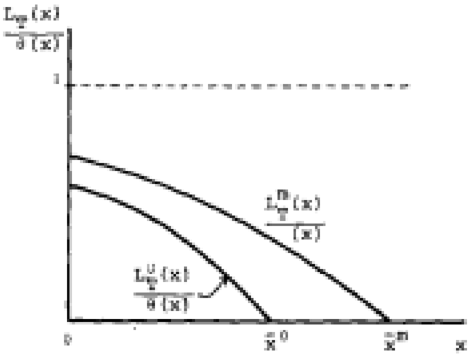

Figure 3
The Proportion of Land Devoted to Roads: A Closed City

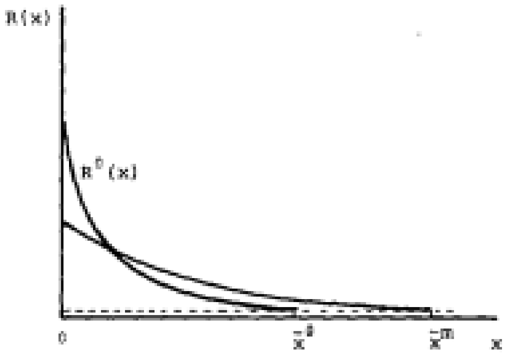

Figure 4
Optimum and Market Rent Profiles: A Closed City

The rent function is plotted in Figure 4. The rent is higher in the optimum city than in the market city near the center but lower near the edge.

As shown in Figure 5, near the CBD the traffic density is higher in the optimum city, which reflects the fact that the road is narrower in the optimum city. Near the edge of the city, however, the traffic density is higher in the market city even though the market city has the wider road, because the optimum city has fewer commuters near the edge simply because the optimum city is smaller.

Table 2 shows the results of the case of $g = 0.5 \times 10^{-8}$, $k = 2$ and $\alpha = 0.2$. The assumption of $k = 2$ implies more acute congestion than in the previous case. This is the reason why the difference in the utility level is greater here. All the qualitative results are the same, however.

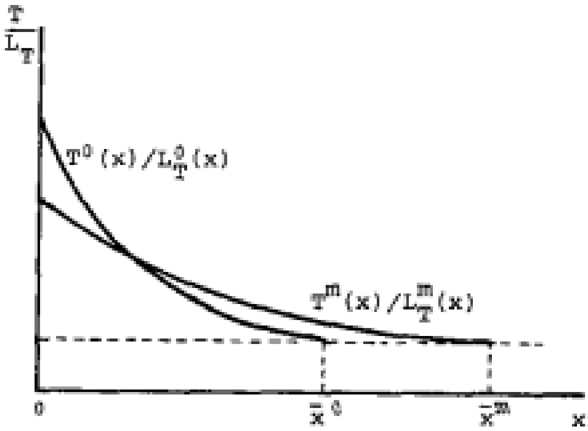

Figure 5
Optimum and Market Traffic Density Functions: A Closed City

3. An Open City

In this section we consider a small, open city where the utility level in the city must equal the level outside the city. This case would be relevant when a planner of a small city is contemplating a long-run policy. In order to isolate problems pertaining to traffic congestion from others, we assume constant returns to scale in production: the aggregate production function of the city is

$$wP,$$

with a constant w. The analysis can be easily extended to the case where the aggregate production function of a city exhibits increasing or decreasing returns to scale.

In a small, open city, the utility level of residents is given and the population of the city becomes an endogenous variable. Therefore, (2.1) is replaced by

$$u(z(x), h(x)) = \bar{u}, \tag{3.1}$$

where $\bar{u}$ is the utility level given for the city.

Only the absentee-landlord case is considered in this section. Absentee landlords receive congestion tolls as well as land rent. The income of a household, then, is given by *w*, and the resource constraint (2.2) no longer holds.

3.1 The Optimum City

As in Chapter I, we maximize the net product of the city after the cost of maintaining the given utility level of residents. Thus our problem is one of maximizing

$$\int_0^{\bar{x}} \{[w - z(x) - t(x)]L_H(x)/h(x) - R_a\theta(x)\}dx\,, \qquad (3.2)$$

subject to the constraints (1.1), (1.6), (1.7), (1.9), (1.10), and (3.1). The Hamiltonian is

$$\Phi = [w - z(x) - t(x)]\frac{L_H(x)}{h(x)} - R_a\theta(x) - \lambda(x)\frac{L_H(x)}{h(x)} + \eta(x)g(T(x), L_T(x))\,, \qquad (3.3)$$

and the Lagrangian for the problem of maximizing the Hamiltonian under the constraints (1.1) and (3.1) is

$$\Psi = \Phi + \nu(x)[u(z(x), h(x)) - \bar{u}] + \mu(x)[\theta(x) - L_H(x) - L_T(x)], \qquad (3.4)$$

where $\lambda(x)$ and $\eta(x)$ are respectively adjoint variables associated with differential equations (1.9) and (1.6). $\nu(x)$ and $\mu(x)$ are Lagrange multipliers for (3.1) and (1.1).

If we define $R(x) = \mu(x)$ and $\tau(x) = t(x) + \lambda(x)$, the first order conditions become, after simple manipulations:

$$\tau'(x) = g + Tg_T \qquad (3.5)$$

$$w = z + Rh + \tau \qquad (3.6)$$

$$-Tg_{L_T}(T, L_T) = R(x) \qquad (3.7)$$

$$\frac{u_h}{u_z} = R(x) \qquad (3.8)$$

$$\tau(0) = 0 \qquad (3.9)$$

$$R(\bar{x}) = R_a\,. \qquad (3.10)$$

It can be seen immediately that these equations coincide with (2.15), (2.18), (2.19), (2.21), (2.14) and (2.22) obtained in a closed city if *w* is replaced by *y*. Therefore, the difference between open and closed cities lies in the determination of income and utility levels. In a closed city these two variables are determined so as to satisfy the population constraint (2.1) and the resource constraint (2.2), whereas in an open city the utility level is given from outside, and the income level is equal to the marginal productivity of labor, which is also assumed to be given. Note that neither land rent nor congestion tolls are returned to residents in this section.

If the production sector is competitive, the wage will be equal to the marginal productivity of labor. Therefore, if the land is owned by absentee landlords, the optimal solution can be obtained by levying congestion tolls and by constructing roads so as to equate the market rent to the marginal benefit from widening the road.

3.2 The Market City

In the market city, it is assumed that production is carried out competitively. Since we consider only the absentee landlord case, the income of residents is the competitive wage *w*. In the absence of congestion tolls, the commuting costs are given by *t*(*x*). When the market rent is given by $R(x)$, a household faces the budget constraint,

$$w = z(x) + R(x)h(x) + t(x).$$

A household maximizes the utility level under this budget constraint, which yields the first order condition:

$$\frac{u_h}{u_z} = R(x)\,.$$

The maximized utility level must be equal to the given utility level, $\bar{u}$, everywhere in the city.

Roads are built according to the benefit-cost criterion based on market prices:

$$-Tg_L(T, L_T) = R(x) \quad .$$

At the edge of the city, the market rent equals the rural rent:

$$R(\bar{x}) = R_a\,.$$

Again, we can observe that the only differences between closed and open cities are boundary conditions which determine the income level and the utility level: in a closed city the income level is given by (2.23) and the population constraint (2.1) must hold, but in an open city the utility level is fixed at $\bar{u}$ and the income level is also a constant *w*.

3.3. Comparison Between Optimum and Market Cities

The optimum and market cities obtained in the previous sections are compared.

First, since $\tau(0)$ and $t(0)$ are both zero, households at $x = 0$ face the same budget constraint,

$$w = z + R(0)h\,,$$

in both optimum and market cities. In order for the utility levels to be the same, the rents at $x = 0$ must be the same in the two cities:

$$R^0(0) = R^m(0)\,,$$

where superscripts ^{0} and ^{m} respectively denote optimum and market solutions.

Since congestion tolls are levied in the optimum city, it is expected that households pay more transportation costs in the optimum city. If this is true, the rent function has a steeper slope in the optimum city than in the market city and the land rent in the optimum city is lower than that in the market city everywhere in the city except at $x = 0$ where they are equal. Though we have not been able to show this in a general case, it is true if we assume the Cobb-Douglas type utility function (2.27) and the Vickrey type transportation cost function (2.28).[4] The rent profiles in this case are depicted in Figure 6.

The traffic density has the same pattern as the rent function. At $x = 0$, optimum and market cities have the same traffic densities and in the rest of the city the market city has a higher traffic density.

The market city size is bigger than the optimum city size. It can be shown that, in the case of the Cobb-Douglas utility function and the Vickrey transportation cost function, the residential zone is exactly $k+1$ times longer in the market city than in the optimum city. In a closed city, the difference between market and optimum city sizes is not as large as in an open city. This is because the population is fixed in a closed city and the rent at the CBD and the income level are both higher in the optimum city.

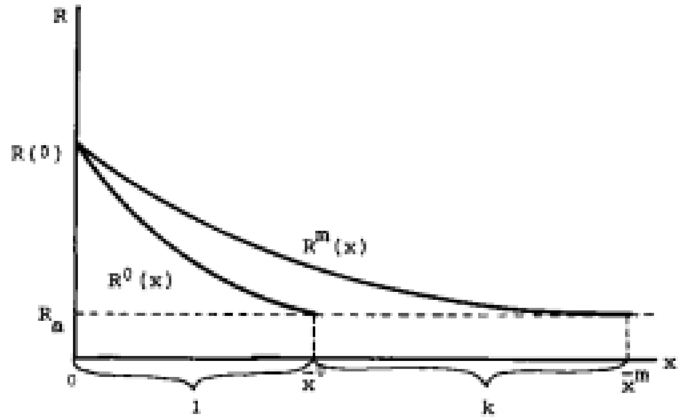

Figure 6
Rent Profiles in Market and Optimum Cities: An Open City

The widths of roads have been calculated for a variety of parameters. In most cases the market city has a wider road than the optimum city though we have found some exceptional cases where the optimum city has a wider road near the center. However, even in such cases the ratio between the total area of roads and the total area of residential land is greater in the market city. One example in which $w = 1, k = 1, h = 10^{-5}$ and $u = -0.364$ is plotted in Figure 7. In this case roads are wider everywhere in the market city.

[4] See Chapter V, Part I of Kanemoto (1977).

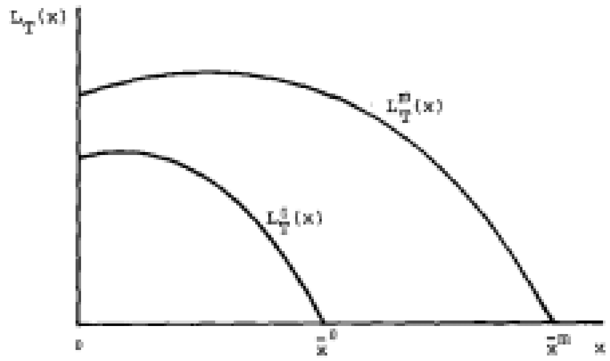

Figure 7
Road Width Functions in Market and Optimum Cities:
An Open City

4. An Economy Consisting of Many Cities

In this section we briefly consider an economy consisting of many cities, under the assumption that the number of cities is a variable. Only the optimum allocation is analysed since the market city size may be indeterminate as shown in Chapter II.

For simplicity, we assume that no one lives in the rural area and that all cities are identical. Then the population of the economy, *P*, the population of a city, *P*, and the number of cities, *n*, must satisfy the relationship

$$P = nP_c \,. \tag{4.1}$$

The boundary condition for *T*(*x*) at $x = 0$ is now

$$T(0) = P_c \,. \tag{4.2}$$

The aggregate production function of a city is

$$F(P_c)\,, \tag{4.3}$$

and we assume increasing returns to scale. The resource constraint (2.2) must be rewritten as

$$\int_0^{\bar{x}} [zL_H/h + Tg(T, L_T) + R_a\theta]dx \le F(P_c)\,. \tag{4.4}$$

Now, our problem is one of maximizing the common utility level, *u*, subject to the resource constraint (4.4), the traffic flow constraint (1.9), the equal utility constraint (2.4), the land constraint (1.1), the population constraint (4.1), and the boundary conditions for *T*(*x*), (4.2) and (1.10). The Lagrangian for this problem is

$$\Lambda = u + \delta n\left[F(P_c) - \int_0^{\bar{x}} (zL_H/h + Tg + R_a\theta)dx \right]$$
$$+ n\int_0^{\bar{x}} \lambda(x)[-L_H/h - T'(x)]dx$$
$$+ n\int_0^{\bar{x}} \nu(x)[u(z,h) - u]dx + n\int_0^{\bar{x}} \mu(x)(\theta - L_H - L_T)dx$$
$$+ \gamma(nP_c - P) + \varepsilon n[P_c - T(0)]. \quad (4.5)$$

If we define

$$\tau(x) \equiv \frac{1}{\delta}(\lambda(x) - \lambda(0)) \quad (4.6)$$

and

$$R(x) \equiv \mu(x)/\delta\,, \quad (4.7)$$

the first order conditions become, after some rearrangements,

$$F'(Pc) = z(x) + R(x)h(x) + \tau(x) \qquad 0 \le x \le \bar{x}\,, \quad (4.8)$$

and (2.14), (2.15), (2.19), (2.21), and (2.22). The only new condition is (4.8), which means that a worker is paid the value of marginal productivity of labor.

The condition can be related to the results in Chapter II. Multiplying (4.8) by $N(x)$ and integrating from 0 to $\bar{x}$ yields

$$P_c F'(P_c) = \int_0^{\bar{x}} [z(x)N(x) + \tau(x)N(x) + R(x)L_H(x)]dx\,. \quad (4.9)$$

The resource constraint (4.4) holds with equality, and the total transportation costs are the same regardless of how costs at different radii are added,

$$\int_0^{\bar{x}} Tgdx = \int_0^{\bar{x}} tNdx\,.$$

We therefore have

$$F(P_c) = \int_0^{\bar{x}} [z(x)N(x) + t(x)N(x) + R_a\theta(x)]dx\,. \quad (4.10)$$

Subtracting (4.8) from (4.9) yields

$$[F(P_c) - P_c F'(P_c)] + \int_0^{\bar{x}} [R(x)L_H(x) - R_a\theta(x)]dx$$
$$+ \int_0^{\bar{x}} [\tau(x) - t(x)]dx = 0. \quad (4.11)$$

The first square bracket is the profit from production, which is negative because of increasing returns to scale; the second term is the net rent revenue after the payment of the rural rent; and the third term the total congestion toll. Thus the loss incurred by a producer equals the sum of the net rent revenue and the total congestion toll. This is more general than the result in Chapter II, which states that the operating loss of a producer equals the total differential rent, or the market rent minus the rural rent. Notice also the similarity with the results of Section 2 of Chapter III which considers congestion in the consumption of local public goods.

If transportation technology has constant returns to scale, the total congestion toll equals the land rent on the road. In this case (4.11) is equivalent to the result obtained in Chapter II,

$$\left[F(P_c) - P_c F'(P_c)\right] + \int_0^{\bar{x}} \left[R(x) - R_a\right]\theta(x)dx = 0: \tag{4.12}$$

the loss of a producer equals the total differential rent.

Notes

Traffic congestion has usually been analyzed in nonspatial frameworks. Strotz (1965) extended the usual analysis to a spatial model in which a city is divided into a finite number of homogeneous rings. He characterized the optimal solution and showed that the optimal solution requires congestion tolls. He also showed that congestion tolls exceed or less than expenditure on roads if transportation technology has decreasing or increasing returns to scale respectively.

Solow and Vickrey (1971) formulated a model of a long narrow city in which distance is a continuous variable. They solved the problem of minimizing transportation costs using calculus of variation. Mills and de Ferranti (1971) consider a similar transportation-cost-minimization problem in a circular city Livesey (1973) and Sheshinski (1973) extended their model to analyze land use within the CBD. Legey, Ripper and Varaiya (1973) extended the model to include capital. They also introduced the market city where roads are built according to the benefit-cost criterion based on market prices and compared optimum and market cities. It was shown that the market city is more dispersed than the optimum city.

All these papers considered closed cities where the tot product (or the total population) of the city was fixed. Kanemoto (1975) introduced an open city where the city faces fixed export price.

None of the above models allow for substitution between land and other factors. Dixit (1973), Oron, Pines and Sheshinski (1973) and Riley (1974) considered traffic congest in a model which allows for the choice of housing lot size and therefore substitution between land and other goods. Indepedent of our work, Robson (1976) compared optimum and market cities in the same model as ours. He considered the case of $\alpha = 1/2$ in the utility function (2.27). Though calculations are easiest in this case, the assumption implies that all households spend half of their incomes-after-commuting-costs on land which is quite unrealistic.

Kanemoto (1976) considered a production city with substitutability between labor and land in an open city framework. The results are parallel to those in section 3 on the open city.

REFERENCES

Dixit, A., (1973), "The Optimum Factory Town, " *The Bell Journal of Economics and Management Science* 4, 637-651.

Kanemoto, Y., (1975), "Congestion and Cost-Benefit Analysis in Cities, " *Journal of Urban Economics* 2, 246-264.

Kanemoto, Y., (1976), "Optimum, Market and Second-Best Land Use Patterns in a von Thünen City with Congestion, " *Regional Science and Urban Economics* 6, 23-32.

Kanemoto, Y., (1977), *Theories of Urban Externalities*, Ph.D. thesis, Cornell University.

Legey, L., M. Ripper and P. Varaiya, (1973), "Effect of Congestion on the Shape of a City, " *Journal of Economic Theory* 6, 162-179.

Livesey, D.A., (1973), "Optimum City Size: A Minimum Congestion Cost Approach, " *Journal of Economic Theory* 6, 144-161.

Mills, E.S. and D.M. de Ferranti, (1971), "Market Choices and Optimum City Size, " *American Economic Review* 61, 340-345.

Miyao, T., (1978), "A Note on Land-Use in a Square City, " *Regional Science and Urban Economics* 8, 371-379.

Oron, Y., D. Pines and E. Sheshinski, (1973), "Optimum vs. Equilibrium Land Use Patterns and Congestion Toll, " *The Bell Journal of Economics and Management Science* 4, 619-636.

Riley, J., (1974), "Optimal Residential Density and Road Transportation, " *Journal of Urban Economics* 1, 230-249.

Robson, A., (1976), "Cost-Benefit Analysis and the Use of Urban Land for Transportation, " *Journal of Urban Economics* 3, 180-191.

Sheshinski, E., (1973), "Congestion and the Optimum City Size, ” *American Economic Review* 63, 61-66.

Solow, P.M. and W.S. Vickrey, (1971), "Land Use in a Long Narrow City, " *Journal of Economic Theory* 3, 430-447.