CHAPTER VI

NEIGHBOURHOOD EXTERNALITIES AND A CUMULATIVE DECAY PROCESS

Whether we like it or not, people often believe they suffer external costs from the presence of some other type of people in their neighbourhood: the rich may fear heavier taxes if poorer households live in the same municipality; whites may not like to live close to blacks; Greeks may believe that their daughters are not safe if there are too many scots in a neighbourhood; and so on. Whether real or imaginary, such externalities raise many issues, some of which are more political or moral than economic. One of the fundamental issues that arise in the context of externalities between different races is whether we approve preferences of individuals who are racially prejudiced: some societies do not, and force individuals to act against their preferences. A typical example is the "busing" regulation in American cities, where school children in a racially segregated area are "bused" to a school at a distant location in order to have racially mixed schools.

Although these issues are extremely important, we concentrate on the economic consequences of the externalities and avoid moral or political judgements. We also restrict ourselves to what might be termed passive discrimination: the well being of discriminators is affected by the locational decisions of others, but discriminators are unable to influence the decisions of others. The reader must be aware that the problem analyzed in this chapter has other important aspects.

We first examine the stability of spatial residential patterns. We find that externalities introduce a tendency toward segregation by type: individuals who suffer an externality from the presence of individuals of another group tend to cluster together to avoid the externality.

We next consider a special kind of a dynamic problem which arises in a city with externalities between different types of households. This analysis is motivated by the experience of American cities in 1960's and 70's. American cities have experienced extensive migration of the middle class households from central cities to the suburbs. Explanations of this phenomenon can be roughly classified into the following two types. The first type sees the migration as an equilibrium process. As the income level rises and commuting costs fall due to technological progress in transportation, the population density gradient becomes flatter in equilibrium. The population in the suburbs, therefore, increases relative to that in central cities. The population increase in the suburbs consists of wealthier families because, for a variety of reasons, richer families have a tendency to live farther from the center.[1]

The second type focuses on the deterioration of central cities that accompanied the out-migration of the middle class. This type explains the process as one of cumulative decay: the deterioration of central cities drives out wealthier residents and so lowers per capita income, and the reduction of per capita income leads to further deterioration. The central city deteriorates cumulatively until it eventually reaches a new equilibrium state. The process of middle class out-migration is thus viewed as a disequilibrium rather than equilibrium process.

In our treatment the decay process appears as a problem of the stability of the boundary between rings of different types of households. When the previously stable boundary becomes unstable as a result of a change in some exogenous factor, a rapid movement of the boundary occurs. The shift to a new stable equilibrium can be interpreted as the cumulative decay process: an increase of one type of households increases the external costs for the other type, causing them to move away and inducing a further increase of the first type.

In section 1 we formulate a model with two types of households, one of which receives a higher income than the other, and also suffer an external cost from the presence of the other. Set up this way, the model can be used to explore the spatial behaviour of 'rich' and 'poor'. Stability of different spatial patterns is examined in section 2. In section 3 we analyze stability of the boundary between the two types, allowing for migration into and out of the city. The possibility of a cumulative process is considered in section 4 and several examples are examined in section 5.

1. The Model

Consider a single-centered city whose residents consist of two types of households that we can call discriminators and nondiscriminators. Discriminators suffer external diseconomy if they live close to nondiscriminators.

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[1] For example, since there are more newer houses in the suburbs, the quality of housing is better in the suburbs. A trade-off between commuting costs and housing also works in favour of the suburban locations of richer households, as seen in Chapter I.

In contrast to our method in previous chapters, we assume that the city stands ready built: houses with certain qualities and lot sizes are already built in the city and the characteristics of houses do not change during the time interval relevant to our analysis.

It is not difficult to relax this assumption and consider the case of malleable housing capital: although the analysis becomes quite tedious, the results are basically the same. The present formulation is preferable because housing capital is in fact quite durable and we are concerned with short-run phenomena. The only serious problem arises at the boundary of the city, where new houses must be built when the city expands. Since we assume that houses are readily available even outside the current boundary of the residential zone, the expansion of the boundary occurs instantaneously in our model. In reality, however, new construction takes time and our results should not be taken too literally. We discuss the problem in the end of section 4.

$h(x)$ denotes the services provided by a house and lot at distance x from the center. Since houses are usually larger farther from the center,

$$h'(x) > 0. \qquad (1.1)$$

Note that in this chapter $h(x)$ denotes the services from both land and buildings, rather than the lot size as in previous chapters.

There are $N(x)\,\mathrm{dx}$ houses in the ring between x and $x + dx$, where we assume that $N(x)$ does not decrease as distance from the center increases:

$$N'(x) \geq 0. \qquad (1.2)$$

This assumption requires that the width of the residential zone, $L_H(x)$, increases faster than the lot size with distance from the center. It precludes the case of a linear city when the lot size increases with distance.

The opportunity cost of a unit of housing services is assumed to be a constant R_a. In equilibrium the rent at the edge of the city must equal R_a:

$$R(\bar{x}) = R_a. \qquad (1.3)$$

Since we assumed that ready-built houses are standing outside the edge of the city, we may take R_a equal to zero. In order to include other possibilities, however, we do not specify the value of R_a in the following analysis.

We want to know how the two groups of households distribute themselves over the ready-built houses when there is externality between the two groups. For the sake of simplicity, we analyze an externality that operates in only one direction.

Discrimination may in fact be extremely complex, but this assumption leads to useful insights about the effect of discrimination on city form. The discriminators, denoted by superscript d, suffers external diseconomies from the presence of nondiscriminators, denoted by superscript n: the nondiscriminators do not experience any externality. Thus the utility function of a discriminator at *x* is

$$u^d(z^d(x), h(x), A(x)) \quad , \tag{1.4}$$

where $A(x)$ denotes the external diseconomy suffered by the discriminator as a result of living near nondiscriminators, and $z^d(x)$ is the consumption of the consumer good. A nondiscriminator at *x* has a utility function with no externality term:

$$u^n(z^n(x), h(x)) \,. \tag{1.5}$$

We assume positive marginal utilities of the consumer good and housing for both, and a negative marginal utility of the externality for the discriminator:

$$u_z^d(z^d, h, A) > 0 \,, \quad u_h^d(z^d, h, A) > 0 \,, \tag{1.6}$$

$$u_z^n(z^n, h) > 0 \,, \qquad u_h^n(z^n, h) > 0, \tag{1.7}$$

$$u_A^d(z^d, h, A) < 0 \,, \tag{1.8}$$

where the subscripts *z*, h, and *A* denote partial derivatives.

The externality given by a nondiscriminator living at x' to a discriminator at x is $a(|x - x'|)$. The function a ( ) is nonnegative and nonincreasing,

$$a(|x - x'|) \geq 0 \,, \tag{1.9}$$

$$a'(|x - x'|) \leq 0 \,, \tag{1.10}$$

and $|x - x'|$ is the absolute value of $x - x'$. The total external diseconomies received by a discriminator at *x* is the sum of diseconomies generated by all nondiscriminators:

$$A(x) = \int_0^\infty a(|x - x'|) N^n(x') dx' \,, \tag{1.11}$$

where $N^n(x')dx'$ is the population of nondiscriminators between x' and $x' + dx'$. If we imagine that the residential zone is circular, (1.11) implies that a nondiscriminator at the same radius, but on the opposite side of the city, induces a larger externality than one very near by but at a slightly different radius. Although this oddity disappears in a linear city, it may affect the generality of the results that follow.

We can now analyze the city forms arising from discrimination if we specify the budget constraints of discriminators and nondiscriminators. Choosing the case which is most common, and probably therefore most interesting, we assume that discriminators are richer than nondiscriminators. A rich discriminator earns an income y^d and pays the commuting costs $t^d(x)$. The budget constraint is

$$y^d = z^d(x) + R(x)h(x) + t^d(x), \tag{1.12}$$

where $R(x)$ is the rent of a unit amount of housing services at x. A poorer nondiscriminator earns a lower income y^n and pays lower commuting costs $t^n(x)$:

$$y^d > y^n \tag{1.13}$$

$$t^{d\prime}(x) > t^{n\prime}(x), \qquad 0 \le x \le \bar{x}, \tag{1.14}$$

$$t^d(0) = t^n(0) = 0. \tag{1.15}$$

Lower commuting costs for a nondiscriminator may be considered as representing lower time costs. Introducing different transportation costs complicates the analysis slightly, as the discussion of the assumption expressed by equation (1.30) below shows. There are, however, gains in realism and in generality which compensate for the additional complexity. The budget constraint for a nondiscriminator is

$$y^n = z^n(x) + R(x)h(x) + t^n(x). \tag{1.16}$$

We assume that neither a discriminator nor a nondiscriminator owns a house in the city. Our model, therefore, corresponds to the absentee-landlord case in Chapter I, with landlords that do not discriminate.

By spatial arbitrage, all households in each group receive equal utility levels in equilibrium:

$$u^d = u^d(z^d(x), h(x), A(x)), \tag{1.17}$$

$$u^n = u^n(z^n(x), h(x)). \tag{l.18}$$

By the assumption of positive marginal utilities, (1.6) and (1.7), these equations can be uniquely solved for z^d and z^d to obtain demand functions for the consumer good,

$$z^d(x) = z^d(h(x), u^d, A(x)), \tag{1.19}$$

$$z^n(x) = z^n(h(x), u^n), \tag{1.20}$$

where

$$z_h^d(h(x), u^d, A(x)) = -u_h^d / z_z^d < 0, \tag{1.21}$$

$$z_u^d(h(x), u^d, A(x)) = 1/u_z^d > 0, \tag{1.22}$$

$$z_A^d(h(x), u^d, A(x)) = -u_A^d / u_z^d > 0, \tag{1.23}$$

and

$$z_h^n(h(x), u^n) = -u_h^n / u_z^n < 0, \tag{1.24}$$

$$z_n^n(h(x), u^n) = 1/u_z^n > 0. \tag{1.25}$$

Substituting (1.19) and (1.20) into (1.12) and (1.16) respectively, we obtain the *bid rent functions*:

$$\begin{aligned} R^d(x) &= \frac{1}{h(x)}[y^d - z^d(h(x), u^d, A(x)) - t^d(x)] \\ &\equiv R^d[I^d(x), u^d, h(x), A(x)], \end{aligned} \tag{1.26}$$

$$\begin{aligned} R^n(x) &= \frac{1}{h(x)}[y^n - z^n(h(x), u^n) - t^n(x)] \\ &\equiv R^n[I^n(x), u^n, h(x)], \end{aligned} \tag{1.27}$$

where

$$I^d(x) \equiv y^d - t^d(x), \tag{1.28}$$

$$I^n(x) \equiv y^n - t^n(x). \tag{1.29}$$

The bid rent functions in this chapter are slightly different from those in other chapters, since $h(x)$ appears in the bid rent functions. A household must take the amount of housing services as given and the only variable a household can choose is the location of a house. It is important to notice that this implies the marginal rate of substitution between housing and the consumer good need not equal the bid rent.

Since the externality $A(x)$ depends on how the nondiscriminators are distributed over space, we must know the locational patterns of the nondiscriminators to obtain the bid rent of the discriminator. The bid rent function of the discriminators, however, influences the spatial distribution of the nondiscriminators. This spatial interrelationship is the only complication in our model.

The following assumption plays a crucial role in determining the stable residential pattern:

$$R_I^n I^n{}'(x) + R_h^n h'(x) = t^n{}'(x) + z_h^n(h(x), u^n)h'(x)$$

$$< t^d{}'(x) + z_h^d(h(x), u^d, A(x))h'(x)$$

$$= R_I^d I^d{}'(x) + R_h^d h'(x), \qquad (1.30)$$

for any relevant range of x, A, u^d, and u^n. This assumption is made to ensure that, if the rich did not discriminate, they would have a flatter bid rent curve and live farther from the center than the poor, as in Chapter I.

From (1.21) and (1.24), we can rewrite (1.30) as

$$\frac{u_h^d(z^d(x), h(x), A(x))}{u_z^d(z^d(x), h(x), A(x))} - \frac{u_h^n(z^n(x), h(x))}{u_z^n(z^n(x), h(x))}$$

$$> \frac{1}{h'(x)}[t^d{}'(x) - t^n{}'(x)] > 0, \qquad (1.31)$$

where the last inequality is obtained from (1.14). The condition can now be interpreted in terms of two opposing forces. First, since discriminators have higher transportation costs, they tend to live closer to the center. Second, if they have a higher marginal rate of substitution between housing and the consumer good than nondiscriminators - if they are willing to give up more of the consumer good for a marginal increase in housing services -, then there is an opposing tendency for discriminators to live in larger houses farther from the center of the city. Our assumption requires that the latter tendency overwhelm the former.

The difference between the marginal rates of substitution between housing and the consumer good is closely related to the normality of housing. Roughly speaking, condition (1.31) is satisfied if housing is a normal good and the normality is strong enough to offset the greater transportation costs of discriminators.[2]

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[2] This statement is precisely true if we assume a utility function which is separable and can be written

$$u^d(z^d, h, A) = U(u^n(z^d, h), A).$$

Given the above functional form, a discriminator has exactly the same preferences over housing and the consumer good as a nondiscriminator, and the preferences are not affected by the externality. Consider

2. Stability of Spatial Patterns

In the absence of externalities, assumption (1.30) assures that the bid rent of the rich will be flatter than that of the poor, and the poor therefore live closer to the center of the city. It can be shown that when the rich suffer external diseconomies, the pattern is unaffected *if the number of housesper unit distance is constant*. This qualification is required because our externality function (1.11) employs only radial distances. If the number of houses per unit distance increases with distance, the assumption (1.30) must be strengthened.

When the number of houses per unit distance is constant,

$$N'(x) = 0 \quad , \qquad\qquad 0 \leq x \leq \bar{x}. \tag{2.1}$$

We assume there is no *active* discrimination in the housing market: neither discriminators nor landlords try to influence where nondiscriminators live.

To see that only the central location of nondiscriminators is stable, we examine each of the possible configurations. The pattern where both the rich discriminators and the poor discriminators live at a same distance from the center is unstable.

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a hypothetical problem of choosing both h and z under the budget constraint, $I = z + Rh$. Because of the separability, the choice of a discriminator is the same for any level of externality. Moreover, both types behave in exactly the same way, and have the same uncompensated demand function for housing, $\hat{h}(I, R)$. As in (I.2.7), the uncompensated demand functions satisfy

$$\hat{h}_I(I,R) = \frac{u_z^i}{D}\left[u_{hz}^i u_z^i - u_h^i u_{zz}^i\right]$$
$$= \frac{(u_z^i)^3}{D}\frac{\partial(u_h^i / u_z^i)}{\partial z},$$

where

$$D = 2u_{hz}^i u_z^i u_h^i - (u_h^i)^2 u_{zz}^i - (u_z^i)^2 u_{hh}^i \geq 0, \qquad\qquad i = n, d.$$

Now if housing is a normal good, we have

$$\frac{\partial(u_h^n / u_z^n)}{\partial z} > 0.$$

Since $z^d(x) > z^n(x)$ from $y^d > y^n$, this implies that

$$\frac{u_h^d(z^d(x), h(x), A(x))}{u_z^d(z^d(x), h(x), A(x))} = \frac{u_h^n(z^d(x), h(x))}{u_z^n(z^d(x), h(x))} > \frac{u_h^n(z^n(x), h(x))}{u_z^n(z^n(x), h(x))},$$

and if the normality is strong enough, (1.31) is satisfied.

Consider a zone at radius x where both the rich and the poor locate. Under the assumption that there is no active discrimination in the housing market, both must pay the same rent at x, and therefore their bid rents must be equal there. Since, by assumption (1.11), the strength of the externality depends only on the radial distance between a discriminator and all nondiscriminators, any increase in the number of nondiscriminators at x drives down the bid rent of the discriminators by increasing the externality. This induces a further increase of the number of nondiscriminators because the bid rent of the nondiscriminators remains the same, and the nondiscriminators outbid the discriminators. The process continues until the zone is filled with nondiscriminators.[3]

It is convenient to introduce a formula which will tell us the relative levels of bid rents of discriminators and nondiscriminators at x'' if we know their relative positions at x'. Since it is simpler to work with the bid rent on a house and lot, $E(x) = R(x)h(x)$, than with the bid rent per unit amount of housing services, $R(x)$, we rewrite (1.26) and (1.27) as

$$E^d(x) \equiv R^d(x)h(x) = y^d - z^d(h(x), u^d, A(x)) - t^d(x),$$

and

$$E^n(x) \equiv R^n(x)h(x) = y^n - z^n(h(x), u^n) - t^n(x).$$

In order to isolate the effect of the externality, we consider the difference between the slopes of $E^d(x)$ and $E^n(x)$ at x between x' and x'', fixing the level of the externality at $A(x'')$:

$$\begin{aligned} H(x; x'') \equiv\ & z_h^n(h(x), u^n)h'(x) + t^n{}'(x) \\ & -[z_h^d(h(x), u^d, A(x''))h'(x) + t^d{}'(x)] \\ & > 0, \end{aligned} \tag{2.2}$$

where the inequality follows from assumption (1.30). We then obtain

$$\begin{aligned} & [E^d(x'') - E^n(x'')] - [E^d(x') - E^n(x')] \\ & = h(x'')[R^d(x'') - R^n(x'')] - h(x')[R^d(x') - R^n(x')] \end{aligned}$$

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[3] Note that this result crucially depends on our assumption (1.11) that the strength of the externality depends only on the radial distance. It is still an open question whether the result carries over to the case where the externality depends also on circumferential distance from a nondiscriminator.

$$= \int_{x'}^{x''} H(x; x'')dx + J(x', x''), \qquad (2.3)$$

where $J(x', x'')$ captures the effect of the difference in the externality between x' and x'':

$$J(x', x'') = z^d[h(x'), u^d, A(x')] - z^d[h(x'), u^d, A(x'')]\,. \qquad (2.4)$$

From (1.23), $J(x', x'')$ satisfies

$$J(x', x'') \gtreqless 0 \quad \text{as} \quad A(x') \gtreqless A(x'')\,. \qquad (2.5)$$

Now consider the case illustrated in Fig.1 where the zone of nondiscriminators extends from x^* to x^{**}, between two zones of rich discriminators. In equilibrium the bid rent of the two groups must be equal at the two borders, since there is no price discrimination in the housing market. Suppose that two bid rents are equal at the inner boundary, x^*, as in Fig.1. From (2.1) the external diseconomy is the same at two boundaries:

$$A(x^*) = A(x^{**}).$$

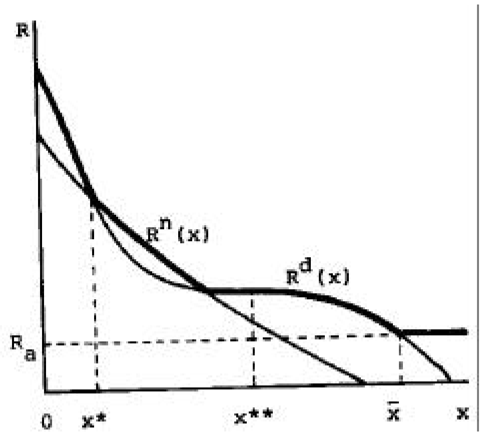

Figure 1
The intermediate location of nondiscriminators

If we set $x' = x^*$ and $x'' = x^{**}$, (2.3) becomes

$$h(x^{**})\left[R^d(x^{**}) - R^n(x^{**})\right] = \int_{x^*}^{x^{**}} H(x; x^{**})dx > 0\,,$$

which implies that the bid rent of discriminators is higher than that of nondiscriminators at the boundary. Therefore, discriminators outbid nondiscriminators in the neighbourhood of the outer boundary and the boundary moves closer to the center.

Thus the intermediate location of nondiscriminators cannot be an equilibrium.

The same reasoning can also be applied to a city which has only two zones, nondiscriminators living in the outer zone and discriminators living in the inner zone.

When nondiscriminators live in more than one zone, denote the borders of nondiscriminators' zone farthest from the center by $x*$ and $x**$. Suppose the two bid rents are equal at $x*$. The externality is stronger at the inner boundary than at the outer boundary, since the inner boundary is closer to other zones of nondiscriminators. Hence, $J(x*, x**)$ is positive and

$$h(x^{**})\left[R^d(x^{**}) - R^n(x^{**})\right]$$
$$= \int_{x^*}^{x^{**}} H(x; x^{**})dx + J(x^*, x^{**}) > 0.$$

This case is not an equilibrium, either.

Finally, consider the case of the central location of nondiscriminators. Let $x*$ be the boundary between the zones of nondiscriminators and rich discriminators as in Fig.2. In equilibrium, the bid rents are the same at the boundary:

$$R^d(x^*) = R^n(x^*) .$$

For any $x' < x*$, we have

$$A(x') \geq A(x^*) .$$

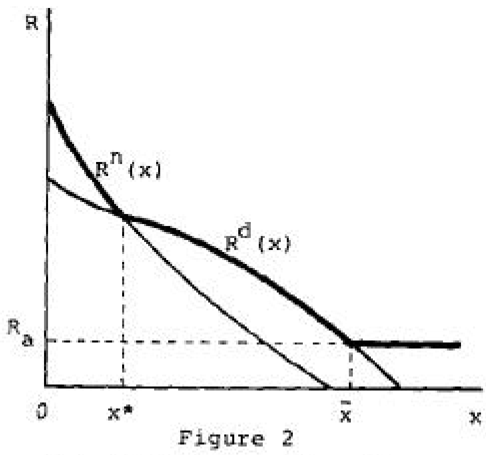

Figure 2
The central location of nondiscriminators

Substituting $x*$ for x'' in (2.3), we obtain

$$-h(x')\left[R^d(x') - R^n(x')\right] = \int_{x'}^{x^*} H(x;x^*)dx + J(x',x^*) > 0\,.$$

Hence, $R^d(x') < R^n(x')$ for any $x' < x^*$ and nondiscriminators outbid discriminators inside the boundary.

At any point, x', outside the boundary, discriminators have a higher bid rent:

$$h(x')\left[R^d(x') - R^n(x')\right] = \int_{x^*}^{x'} H(x;x')dx + J(x'',x') > 0\,,$$

since $A(x') \leq A(x^*)$.

Thus the central location of nondiscriminators is an equilibrium. Since the cases considered here exhaust all the possibilities, the central location of nondiscriminators is the only stable market equilibrium under the assumption (1.30) and (2.1). This result shows that the existence of externalities does not alter the spatial pattern when (1.30) and (2.1) hold. No discrimination is, therefore, necessary to confine nondiscriminators in the central part of the city. Moreover, the external diseconomy makes the segregated pattern more stable since the bid rent curve of discriminators becomes flatter. Note that it is not the strength of the externality that makes the bid rent curve of discriminators flatter, but the fact that externality *diminishes* with distance. It is easy to see that if externality is uniform in the city, no change in the slope of the bid rent curve occurs.

We have shown that passive discrimination of the sort we have modeled can explain the spatial distribution of racial groups, blacks in American cities for example, when the group discriminated against is uniformly poorer than the discriminators. This result does not suggest that *active* discrimination does not exist. Recent studies support the view that there is in fact active discrimination in the housing market of American cities.

If the number of houses per unit distance increases with distance from the center, $N'(x) > 0$, the above result must be modified. In order for the central location of nondiscriminators to be a unique stable configuration, the inequality (1.30) must be strengthened to

$$z_h^d h'(x) + t^d{}'(x) > z_h^n h'(x) + t^n{}'(x) + e\,, \tag{2.6}$$

for some large enough $e > 0$. The problem arises because our externality function (1.11) employs only radial distance. If there are more households per unit distance at larger radii, the externality will be higher at the outer boundary than at the inner boundary, which causes an additional tendency to lower the bid rent of discriminators at the outer boundary. The inequality (1.30), therefore, must be strong enough to offset this effect. In the rest of the chapter, (2.6) is assumed to hold for a sufficiently large e so that the only stable configuration is the central location of nondiscriminators.

3. The Boundary Bid Rent Curves

In section 2 we established the existence of a single boundary between two types of households, with the rich discriminators living farthest from the center. In section 4 we will examine the stability of the boundary between the two zones, but in order to do so we develop an additional concept, the *boundary bid rent curve.* The boundary bid rent curve is the bid rent at x when the boundary is at x.

Assume that the city is open: migration into and out of the city is free and costless. The utility levels of rich discriminators and poor nondiscriminators in the city, u^d and u^n, then equal the corresponding utility levels in the rest of the world, V^d and V^n, respectively. The utility levels, however, are not necessarily fixed. An increase in the population of the city is accompanied by a decrease in the population of the outside world, which causes a rise in the utility level in the outside world because of diminishing returns. We assume that the general utility level of discriminators is a nondecreasing function of the population of discriminators in the city, and, that the same is true for nondiscriminators.[4]

$$u^d = V^d(P^d), \tag{3.1}$$

$$u^n = V^n(P^n), \tag{3.2}$$

where

$$V^{d}{}'(P^d) \geq 0, \tag{3.3}$$

$$V^{n}{}'(P^n) \geq 0, \tag{3.4}$$

and P^d and P^n are respectively the populations of discriminators and

[4] In general, the utility level of discriminators (and also nondiscriminators) depends on the populations of both discriminators and nondiscriminators. For simplicity, we assume that the population of one type has no effect on the utility level of the other type. We make a similar assumption for income levels in (3.5) and (3.6) below.

nondiscriminators in the city. $V^{d}{}'(P^{d})$ and $V^{n}{}'(P^{n})$ are almost zero if there are so many people of each type in the outside world that an additional individual does not cause any significant change in allocation there. Since our formulation implicitly assumes that the population of the city is small enough for an additional individual to matter within the city, this in effect requires that the city is small compared with the rest of the world. Roughly speaking, therefore, $V^{d}{}'(P^{d})$ and $V^{n}{}'(P^{n})$ are zero if the city is small, and increase for cities which are larger relative to the rest of the world.

The income of a city resident also depends on the population of the city. This reflects two factors. First, if prices of products are constant, the wage rate falls due to diminishing returns as the population increases. Second, when the population increases, production expands, which reduces prices of products in the world market. This also causes a decrease in wage rate. We therefore assume that the income of each type of household is a nonincreasing function of the population of that type in the city,

$$y^{d} = y^{d}(P^{d}), \tag{3.5}$$

$$y^{n} = y^{n}(P^{n}). \tag{3.6}$$

where

$$y^{d}{}'(P^{d}) \leq 0, \tag{3.7}$$

$$y^{n}{}'(P^{n}) \leq 0. \tag{3.8}$$

$y^{d}{}'(P^{d})$ and $y^{n}{}'(P^{n})$ are smaller in absolute value in a smaller city, since the effects on the world prices are smaller by the same argument as we applied to the case of $V^{d}{}'(P^{d})$ and $V^{n}{}'(P^{n})$.

Let $x*$ denote the boundary between the zones of discriminators and nondiscriminators. Then

$$P^{n}(x^{*}) = \int_{0}^{x^{*}} N(x)dx, \tag{3.9}$$

$$P^{d}(x^{*}) = \int_{x^{*}}^{\bar{x}} N(x)dx, \tag{3.10}$$

where $\bar{x}$ is determined so that the highest bid rent equals the rural rent, R_a, at the edge of the city.

Now, we express bid rent functions as functions of $x*$ using (3.9) and (3.10). The bid rent, $R^n(x;x^*)$, of a nondiscriminator at x when the boundary is at $x*$ is

$$R^n(x;x^*) = \frac{1}{h(x)}\left\{y^n(P^n(x^*)) - z^n\left[h(x), V^n(P^n(x^*))\right] - t^n(x)\right\}. \tag{3.11}$$

The slope of the bid rent curve is

$$\begin{aligned} R_x^n(x;x^*) &\equiv \frac{\partial}{\partial x} R^n(x;x^*) \\ &= -\frac{1}{h(x)}\left[z_h^n h'(x) + t^n{}'(x)\right] - R^n(x)\frac{h'(x)}{h(x)} \end{aligned} \tag{3.12}$$

The location of the boundary enters this formulation, not because nondiscriminators discriminate, but because the location of the boundary determines P^n, which affects income and utility levels.

The bid rent of discriminators depends in addition on the externality that they suffer from nondiscriminators. The externality received by a discriminator at x is

$$A(x;x^*) = \int_0^{x^*} a(|x - x'|)N(x')dx'. \tag{3.13}$$

The bid rent of discriminators is then

$$\begin{aligned} &R^d(x;x^*) \\ &= \frac{1}{h(x)}\left\{y^d(P^d(x^*)) - z^d\left[h(x), A(x;x^*), V^d(P^d(x^*))\right] - t^d(x)\right\} \end{aligned} \tag{3.14}$$

Its slope is

$$R_x^d(x;x^*) = -\frac{1}{h(x)}\left[z_h^d h'(x) + t^d{}'(x) + z_A^d A_x\right] - R^d(x)\frac{h'(x)}{h(x)}. \tag{3.15}$$

A_x is defined as

$$A_x \equiv \frac{\partial}{\partial x} A(x;x^*) = \int_0^{x^*} \mathrm{sgn}(x - x')a'(|x - x'|)N(x')dx', \tag{3.16}$$

where

$$\mathrm{sgn}(x - x') = \begin{cases} +1 & \text{if} \quad x - x' \geq 0, \\ -1 & \text{if} \quad x - x' < 0. \end{cases}$$

A_x is nonpositive at least when x is greater than or equal to $x*$. The externality, therefore, tends to make the bid rent curve of discriminators flatter. It follows from assumption (1.30) that the bid rent curve of discriminators is flatter than that of nondiscriminators at the boundary:

$$\begin{aligned} R_x^d(x^*;x^*) &= -\frac{1}{h(x^*)}\left[z_h^d h'(x^*) + t^d{}'(x^*) + z_A A_x\right] - R^d(x^*)\frac{h'(x^*)}{h(x^*)} \\ &\geq -\frac{1}{h(x^*)}\left[z_h^d h'(x^*) + t^d{}'(x^*)\right] - R^d(x^*)\frac{h'(x^*)}{h(x^*)} \\ &\geq -\frac{1}{h(x^*)}\left[z_h^n h'(x^*) + t^n{}'(x^*)\right] - R^n(x^*)\frac{h'(x^*)}{h(x^*)} \\ &= R_x^n(x^*;x^*) \end{aligned} \tag{3.17}$$

This confirms the result in the preceding section that nondiscriminators live closer to the center.

At the edge of the city, the bid rent of discriminators must equal the rural rent:

$$R^d(\bar{x};x^*) = R^a, \tag{3.18}$$

which determines $\bar{x}$ as a function of x^* and hence $P^d(x^*)$ in (3.10).

Next, we introduce the concept of the *boundary bid rent curve*, which is the bid rent at x when the boundary is at x. The boundary bid rent curves will play a crucial role in the analysis of a cumulative process. For nondiscriminators it is

$$\hat{R}^n(x) = R^n(x;x), \tag{3.19}$$

and from (3.11) its slope is

$$\begin{aligned} \hat{R}^n{}'(x) &= R_x^n(x;x) + R_{x^*}^n(x;x) \\ &= R_x^n(x;x) - \left[z_u^n v^n{}'(P^n) - y^n{}'(P^n)\right]\frac{N(x)}{h(x)} \\ &\leq R_x^n(x;x), \end{aligned} \tag{3.20}$$

where

$$R_{x^*}^n(x;x^*) \equiv \partial R^n(x;x^*)/\partial x^*.$$

Thus the boundary bid rent curve is steeper than the bid rent curve. An expansion of the boundary is possible only if the population of nondiscriminators increases in the city. This raises the utility level in the outside world and lowers the income level in the city, causing a fall in the bid rent curve. The relationship between the bid rent curve and the boundary bid rent curve is illustrated in Figure3.

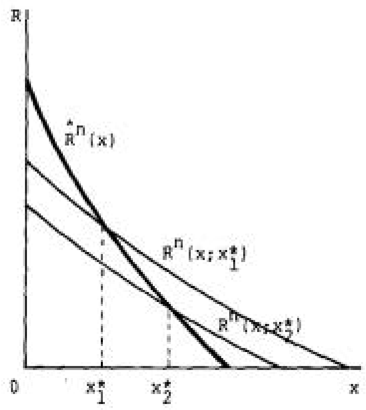

Figure 3
The relationship between the boundary bid rent curve and the bid rent curves

The boundary bid rent of discriminators is

$$\hat{R}^d(x) = R^d(x;x), \tag{3.21}$$

with a slope

$$\hat{R}^{d}{}'(x) = R_x^d(x;x) - \frac{1}{h(x)}\left\{z_A^d A_{x^*}(x;x) + \left[z_u^d v^d{}'(P^d) - y^d{}'(P^d)\right]P^d{}'(x)\right\} \tag{3.22}$$

where

$$A_{x^*}(x;x^*) \equiv \frac{\partial}{\partial x^*} A(x;x^*) \tag{3.23}$$

and hence

$$A_{x^*}(x;x) = a(0)N(x) \geq 0. \tag{3.24}$$

Whether the slope of the boundary bid rent curve of the discriminators is steeper than that of the bid rent curve is not clear. An outward movement of the boundary acts on the bid rent curve of discriminators in two opposing ways. The increased population of nondiscriminators drives up the externality causing the bid rent to fall. As will be shown, however, $P^d{}'(x)$ is usually negative: the population of discriminators decreases as the boundary moves outward, lowering the utility for discriminators in the outside world, increasing their income in the city and tending to cause their bid rent to rise. The boundary bid rent curve of discriminators is therefore either flatter or steeper than the bid rent curve depending on which tendency is stronger.

The first term in the square bracket of (3.22), $z_A^d A_{x^*}(x;x)$, is positive from (3.24), since z_A^d is positive from (1.23). The second term is more complicated. $P^d{}'(x)$ can be obtained by differentiating (3.10) and (3.18).

$$P^d{}'(x) = -\frac{z_A^d\left[N(\bar{x})A_{x^*} + N(x)A_x\right] + N(x)\left[t^d{}'(\bar{x}) + (R_a + z_h^d)h'(\bar{x})\right]}{\left[z_u^d v^d{}'(P^d) - y^d{}'(P^d)\right]N(\bar{x}) + t^d{}'(\bar{x}) + z_A^d A_x(\bar{x};x) + (R_a + z_h^d)h'(x)} \quad . \quad (3.25)$$

The first square bracket on the numerator is positive under the assumption that $N'(x) \geq 0$, since

$$\begin{aligned} &N(\bar{x})A_{x^*}(\bar{x};x) + N(x)A_x(\bar{x};x) \\ &= N(x)\left[a(\bar{x})N(\bar{x}) + \int_0^x a'(|\bar{x}-x'|)(N(x') - N(\bar{x}))dx'\right] \\ &> 0. \end{aligned} \quad (3.26)$$

The first term in the second square bracket of the numerator of (3.25) is positive but the second term may be negative. The second term is zero if the marginal rate of substitution between housing and the consumer good equals the bid rent, which is the case if $h(x)$ can be freely chosen. If houses are newly constructed at the edge of the city, $h(\bar{x})$ may be optimize It is, therefore, plausible to assume that the magnitude of the second term is small. Thus, the numerator tends to be positive.

The first two terms of the denominator are positive. The third term is nonpositive but the magnitude is small since the externality is weak at the edge of the city. The fourth term is also small since $R_a + z_h^d$ is small as argued above. Therefore, the denominator also tends to be positive and $P^d{}'(x)$ is likely to be negative.

The reason for this result is roughly as follows. If the zone of nondiscriminators expands, the city must expand to accommodate the same population of discriminators. Consider the effects on a discriminator at the edge of the city. There are three major effects: commuting costs increase, the strength of the externality increases since there are more nondiscriminators in the city, and the boundary shifts outward to where houses are larger, by the assumption that $h'(x) > 0$. The first two effects tend to lower the utility level of the discriminator, but the direction of the third effect depends on whether houses are larger or smaller than the optimum at the edge of the city. If houses are smaller than the optimum, the third effect tends to raise the utility level. Since the third effect disappears when $h(x)$ is optimized, the first two effects are likely to be dominant, and the utility level declines as x^* increases. This induces emigration of discriminators, resulting in a decrease in the population of discriminators in the city.

4. Stability of the Boundary and a Cumulative Process

Next, we examine stability of the boundary between the zones of rich discriminators and poor nondiscriminators. It is easy to show that, if the boundary bid rent of discriminators is less steep than that of nondiscriminators, the boundary is stable, and if steeper, the boundary is unstable. Consider the situation represented by Fig.4b. The *boundary* is at x^*, and beyond x^* the discriminators outbid the nondiscriminators. The boundary bid rent of the discriminators is steeper, however, as illustrated in Fig.4a. Notice that, if the boundary x^* is to be an equilibrium, the boundary bid rents must be equal there as well as the bid rents.

Figure 4. Stability of Boundaries

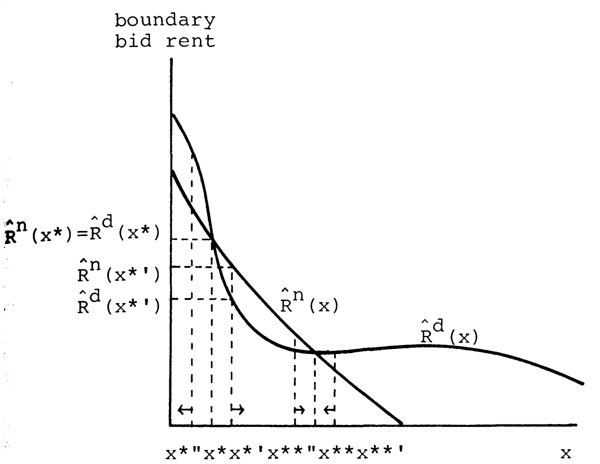

Figure 4a

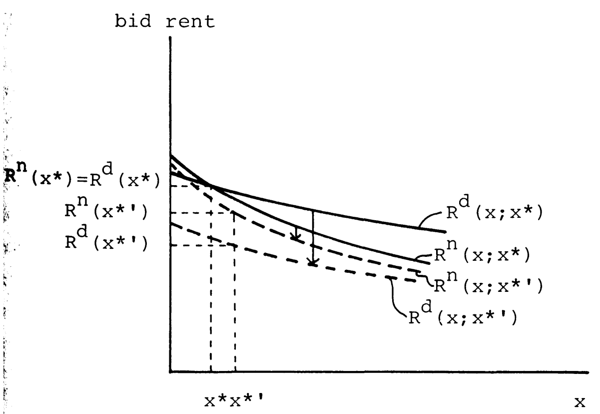

Figure 4b

Now imagine that the boundary shifts outward to $x^{*\prime}$ because of some random disturbances. The bid rent of discriminators falls farther than the bid rent of nondiscriminators, as in Fig.4b. Nondiscriminators outbid discriminators at the new boundary and the boundary moves farther outward. The process continues until the boundary reaches x^{**}.

If the boundary had shifted inward, discriminators would have outbid nondiscriminators, causing the boundary to move inward until it reached the center, x^{*} is therefore unstable. The same argument applied at x^{**} will show that the boundary is stable at that point.

We have seen that the bid rent curve of discriminators is flatter than that of nondiscriminators at the boundary. As shown in the preceding section, the boundary bid rent curve of nondiscriminators is steeper than their bid rent curve, and the boundary bid rent curve of discriminators is flatter than their bid rent curve if the externality is weak. In order to have an unstable equilibrium, therefore, the externality must be strong.

We next examine the condition for an unstable equilibrium more carefully. The difference between the slopes of the two boundary bid rent curves is

$$\begin{aligned} &\hat{R}^{d}{}'(x)-\hat{R}^{n}{}'(x) \\ &\quad =-\frac{1}{h(x)}\{-H(x;x)+z_A^d\hat{A}'(x) \\ &\quad\quad +[(z_u^d V^{d}{}'-y^{d}{}')p^{d}{}'(x)-(z_u^n v^{n}{}'-y^{n}{}')N(x)]\} \end{aligned} \quad , \tag{4.1}$$

where

$$\begin{aligned} \hat{A}'(x) &\equiv \frac{d}{dx}A(x;x) \\ &=a(x)N(x)+\int_0^x a'(|x-x'|)(N(x')-N(x))dx'>0 \end{aligned} \quad . \tag{4.2}$$

The first and third terms in the brace of (4.1) are negative, and the second term positive. Therefore, if the second term is greater than the absolute value of the sum of the first and third terms, the boundary bid rent curve of discriminators is steeper than that of nondiscriminators, and the boundary is unstable. This is more likely to occur if

(a) $H(x;x)$ is smaller: the tendency of the poor to live closer to the center in the absence of the externality is smaller;

(b) $(z_u^d V^{d}{}'-y^{d}{}')P^{d}{}'(x)-(z_u^n V^{n}{}'-y^{n}{}')N(x)$ is smaller: the city is smaller in comparison with the rest of the world;

(c) z_A^d is bigger: the marginal disutility of the external diseconomy is larger;

(d) $a(x)$ is bigger, which is true if x is smaller, that is, the boundary is closer to the center, or if the externality diminishes less rapidly with distance.[5]

Now consider a historical process in which bid rent shift up due to some exogenous factors such as technological progress. We assume that the bid rent of nondiscriminators rises more rapidly than that of discriminators. This assumption does not necessarily mean that the income of nondiscriminators rises more rapidly than that of discriminators. Even if the income of discriminators were to rise faster than that of nondiscriminators, the bid rent of nondiscriminators might rise faster if the utility level of nondiscriminators attainable in the rest of the world was increasing more slowly. To make our analysis easier, we fix the boundary bid rent of discriminators and allow the boundary bid rent of nondiscriminators to rise over time.

Since the boundary bid rent curve depends on the choice of utility and transportation cost functions and on other parameters of the model, we cannot say much, *a priori*, about its shape. Instead, we illustrate a few examples. If the boundary bid rent curve of nondiscriminators is steeper than

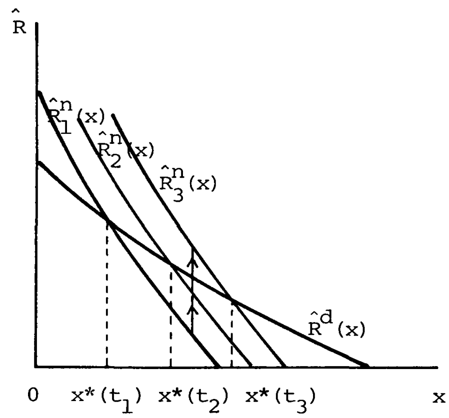

Figure 5. No Cumulative Process

that of discriminators everywhere in the city, we obtain Fig.5. In this case, $x^*(t_1), x^*(t_2)$ and $x^*(t_3)$ are all stable and the boundary gradually shifts outward as the bid rent of nondiscriminators rises.

[5] The integral in (4.2) is greater when $N'(x)$ is greater. However, if $N'(x)$ is large, H must be large enough to insure the central location of nondiscriminators, and the net effect is uncertain.

Figure 6 depicts the case where the boundary bid rent curve of discriminators is steeper at the center. At time t_1, $x^*(t_1)$ is an unstable equilibrium and $x^{**}(t_1)$ is a stable equilibrium. If the boundary were to the right of $x^*(t_1)$, it would move to a stable equilibrium at $x^{**}(t_1)$. With the boundary initially at $x=0$, however, no nondiscriminator would enter the city until time t_2, when the boundary bid rent at $x=0$ of nondiscriminators rose as high as that of discriminators. Then any small perturbation would induce a sudden outward shift of the boundary to $x^{**}(t_2)$. Thus a very rapid movement of discriminators to the suburbs occurs after the first nondiscriminator enters the city.

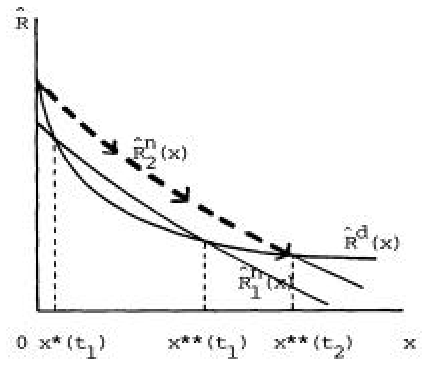

Figure 6
A Cumulative Process at the Center

Finally, consider the case where the boundary bid rent curve of discriminators is flatter than that of nondiscriminators near the center but becomes steeper at some point as in Figure 7. In this case, the boundary gradually moves outward until the bid rent of nondiscriminators becomes tangent to that of discriminators, and then jumps to x^{**}.

Figure 7b illustrates the corresponding bid rent curves. The rapid shift of the boundary is accompanied by a downward shift of both bid rent curves. The bid rent of nondiscriminators must fall because the population of nondiscriminators in the city rises, resulting in an increase in the external utility level by (3.4). Since $u^n = v^n(p^n)$, the utility level of nondiscriminators in the city must also rise, and for utility levels to rise rents must fall. Similarly, since the shifting boundary would usually drive out some discriminators, the utility level of discriminators falls, and rents are likely to rise. Paradoxically, then, the so-called deterioration of the city center may be desirable in terms of income distribution.

In the inner part of the zone of discriminators, the increased externality leads to a fall in the bid rent. In the outer part, however, the rent will usually rise.

The cumulative decay process analyzed by Baumol (1972a, b) and by Oates, Howrey, and Baumol (1971) can be viewed as a rapid movement of the boundary of the sort described in this chapter. If a small increase in the number of nondiscriminators lowers the utility level of discriminators, the discriminators move out to the suburbs, leading to a further deterioration of central cities. This process occurs only if the rent does not fall sufficiently to compensate discriminators for the increase in the external diseconomy, or in our model only if the boundary bid rent curve of nondiscriminators is flatter than that of discriminators.

As discussed in section 1, the fact that the cumulative process is instantaneous in our model depends on the assumption that houses are readily available even outside the current boundary of the city. In reality, however, houses cannot be constructed immediately, and the cumulative process may take

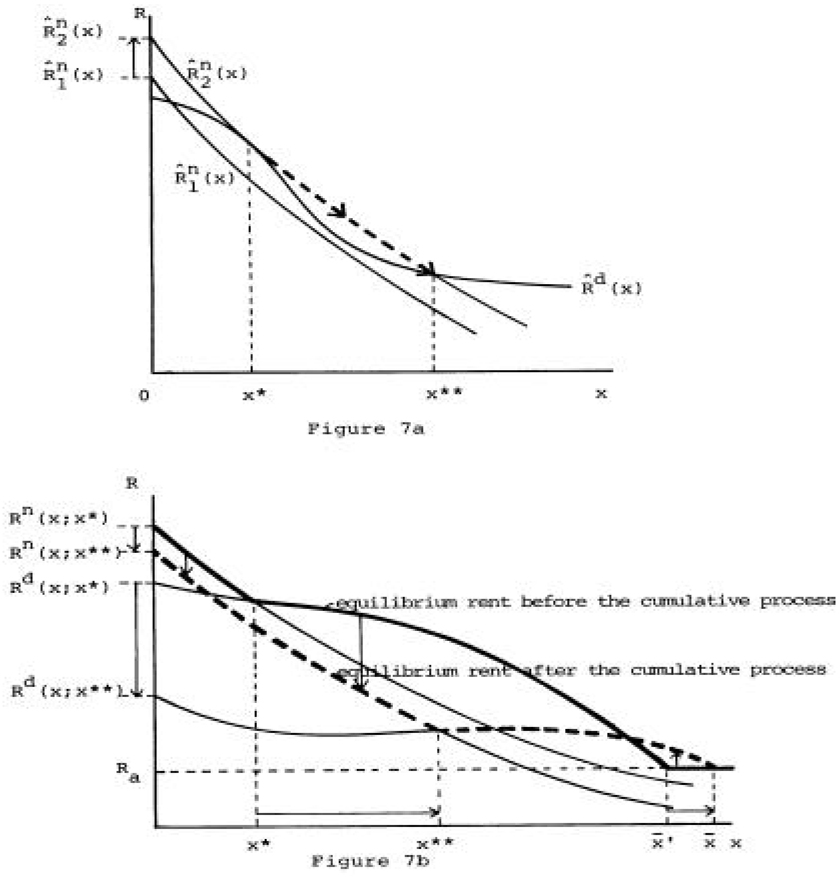

Figure 7. A Cumulative Process at a Later Stage

quite a long time. It is not the rate of change that characterizes a cumulative decay process, however. The process is cumulative if it can be seen as a disequilibrium process moving towards a new equilibrium, like the boundary shift from x^* to x^{**} in

Figure 7, rather than an equilibrium process.

Notes

Baumol formalized a process of cumulative urban deterioration in section 7 of his paper (1972a). Policy implications of his model were further analyzed by Baumol (1972b) and Oates, Howrey and Baumol (1971). The process of decay is described by two difference equations. One equation embodies the mechanism through which an increase in deterioration leads to a reduction in income per capita in the subsequent period as a consequence of induced emigration to the suburbs, while the other equation describes how the fall in income induces further deterioration. For a suitable set of parameters, these equations obviously have a solution which converges to an equilibrium point and the process toward an equilibrium can be viewed as a cumulative process of urban deterioration.

The weakness of this argument is that individual behaviour and market adjustments are not explicitly considered. For example, we immediately face the following question. Why does the land rent in the central city not fall to keep the wealthier people in the center? If the land rent falls sufficiently, wealthier people will remain even in the deteriorated central city. For a cumulative deterioration process to occur, therefore, something must prevent the land rent from falling enough.

Obviously the rent cannot fall below zero. If it reaches zero, therefore, a cumulative process occurs. In this case deterioration results in vacant houses. Alternatively, poorer households might support the rent. This case can occur in two ways. One is through an increase in *per capita* housing demand by the poorer households and the other is through migration of the poorer households from other areas. Our model in this chapter formalizes the latter case.

Kanemoto (1978) considered the same problem in a simpler model with three discrete regions: the city center, the suburbs, and the rest of the world. The paper explores the case of fiscal burden and the case where one type receives an external *economy* from the other type while generating an external diseconomy. The model in this chapter can easily be extended to include these cases.

We chose not to formulate an explicit dynamic adjustment model because exposition would become tedious, and because the basic results can be explained heuristically, as done in this chapter. Schelling was the first to formulate dynamic models of segregation in the housing market in his papers (1971) and (1972), following his earlier work (1969). Miyao (1978a) extended his analysis, explicitly including the individual choice of space and location within a city. Kanemoto (1978) also considered a model of dynamic adjustment. Miyao (1978b) considered the same problem in the framework of a probabilistic model of locational choice. These analyses correspond to that in section 2.

Yellin (1974), Rose-Ackerman (1975), (1977), Yinger (1976a, b), and Courant and Yinger (1977) provide static analyses of an externality between different types of households. Yellin has the most general formulation of the externality which we adopted in this chapter.

Although we did not use any results from mathematical theory of catastrophe, our analysis may be cast in that framework. In section 4 we examined how the phase portrait changes as various parameters change. A cumulative process occurs at what is called in catastrophe theory a bifurcation point, where a basic change in the phase portrait occurs: a stable equilibrium becomes an unstable equilibrium.

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